# THE CALLIAS INDEX FORMULA REVISITED

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Abstract. We revisit the Callias index formula in connection with supersymmetric Dirac-type operators $H$ of the form

$$H = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$$

in odd space dimensions $n$, originally derived in 1978, and prove that

$$\begin{aligned} \operatorname{ind}(L) = \left(\frac{i}{8\pi}\right)^{(n-1)/2} \frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\dots i_n} \\ \times \int_{\Lambda S^{n-1}} \operatorname{tr}_{\mathbb{C}^d}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x))x_{i_n}\, d^{n-1}\sigma(x), \end{aligned} \tag{0.1}$$

where

$$U(x) := |\Phi(x)|^{-1}\Phi(x) = \operatorname{sgn}(\Phi(x)), \quad x \in \mathbb{R}^n.$$

Here the closed operator $L$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ is of the form

$$L = \mathcal{Q} + \Phi,$$

where

$$\mathcal{Q} := Q \otimes I_d = \left(\sum_{j=1}^{n} \gamma_{j,n}\partial_j\right) I_d,$$

with $\gamma_{j,n}$, $j \in \{1,\dots,n\}$, elements of the Euclidean Dirac algebra, such that $n = 2\widehat{n}$ or $n = 2\widehat{n}+1$. Here $\Phi$ is identified with $I \otimes \Phi$, satisfying

$$\Phi \in C_b^2\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big), \quad d \in \mathbb{N},$$
$$\Phi(x) = \Phi(x)^*, \quad x \in \mathbb{R}^n,$$

there exists $c > 0$, $R \geqslant 0$ such that

$$|\Phi(x)| \geqslant cI_d, \quad x \in \mathbb{R}^n \backslash B(0,R),$$

and there exists $\varepsilon > 1/2$ such that for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| < 3$, there is $\kappa > 0$ such that

$$\|(\partial^\alpha\Phi)(x)\| \leqslant \begin{cases} \kappa(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa(1+|x|)^{-1-\varepsilon}, & |\alpha| = 2, \end{cases} \quad x \in \mathbb{R}^n.$$

These conditions on $\Phi$ render $L$ a Fredholm operator, and to the best of our knowledge they represent the most general conditions known to date for which Callias' index formula (0.1) has been derived.

We also consider a generalization of the index formula (0.1) to certain classes of non-Fredholm operators $L$ for which (0.1) represents its (generalized) Witten index (based on a resolvent regularization scheme).

*Date*: January 19, 2016.

2010 *Mathematics Subject Classification.* Primary 47A53, 47F05; Secondary 47B25.

*Key words and phrases.* Fredholm index, Witten index, resolvent regularization, Dirac-type operators, Callias index formula.

To appear in *Springer Lecture Notes in Mathematics*.

Contents

## 1. Introduction

If pressed to describe the contents of this manuscript in a nutshell, one could say we embarked on an attempt to settle the Callias index formula, first presented by Callias [22] in 1978, with the help of functional analytic methods. While we tried at first to follow the path originally envisaged by Callias, we soon had to deviate sharply from his strategy of proof as we intended to derive his index formula under more general conditions on the potential $\Phi$ in the underlying closed operator $L$ (see (1.4)), but also since several of the claims made in [22] can be disproved.

Before describing the need to reconsider Callias' original arguments, and before entering a brief discussion of new developments in the field since 1978, it may be best to set the stage for the remarkable index formula that now carries his name.

For a given spatial dimension $n \in \mathbb{N}$, we denote the elements of the *Euclidean Dirac algebra* (cf. Appendix A for precise details) by $\gamma_{j,n}$, $j \in \{1, \dots, n\}$. One recalls in this context that for $n = 2\widehat{n}$ or $n = 2\widehat{n}+1$ for some $\widehat{n} \in \mathbb{N}$, $\gamma_{j,n}$ satisfy

$$\gamma_{j,n}^* = \gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}} \times 2^{\widehat{n}}}, \quad \gamma_{j,n}\gamma_{k,n} + \gamma_{k,n}\gamma_{j,n} = 2\delta_{jk} I_{2^{\widehat{n}}}, \quad j,k \in \{1, \dots, n\}. \tag{1.1}$$

With the elements $\gamma_{j,n}$ in place, one then introduces the constant coefficient, first-order differential operator $Q$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ by

$$Q \coloneqq \sum_{j=1}^{n} \gamma_{j,n}\partial_j, \quad \operatorname{dom}(Q) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}}, \tag{1.2}$$

with $H^m(\mathbb{R}^n)$, $m \in \mathbb{N}$, the standard Sobolev spaces. One notes in passing that

$$Q^2 = \Delta I_{2^{\widehat{n}}}, \quad \operatorname{dom}(Q^2) = H^2(\mathbb{R}^n)^{2^{\widehat{n}}}. \tag{1.3}$$

Next, let $d \in \mathbb{N}$ and assume that $\Phi \colon \mathbb{R}^n \to \mathbb{C}^{d \times d}$ is a $d \times d$ self-adjoint matrix with entries given by bounded measurable functions. We introduce the operator $L$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ via

$$L \colon \begin{cases} H^1(\mathbb{R}^n)^{2^{\widehat{n}}d} \subseteq L^2(\mathbb{R}^n)^{2^{\widehat{n}}d} \to L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}, \\ \psi \otimes \phi \mapsto \left(\sum_{j=1}^n \gamma_{j,n}\partial_j\psi\right) \otimes \phi + (x \mapsto \psi(x) \otimes \Phi(x)\phi)\,. \end{cases} \tag{1.4}$$

Given (1.2), we shall abreviate

$$\mathcal{Q} \coloneqq Q \otimes I_d = \left(\sum_{j=1}^{n} \gamma_{j,n}\partial_j\right) I_d, \tag{1.5}$$

and, with a slight abuse of notation, employ the symbol $\Phi$ also in the context of the operation

$$\Phi \colon \psi \otimes \phi \mapsto (x \mapsto \psi(x) \otimes \Phi(x)\phi)\,, \tag{1.6}$$

(see our notational conventions to suppress tensor products whenever possible, collected in Section 2 and in Remark 2.1). Thus, we may write,

$$L = \mathcal{Q} + \Phi. \tag{1.7}$$

The associated (self-adjoint ) supersymmetric Dirac-type operator $H$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d} \oplus L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ is then of the form

$$H = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}. \tag{1.8}$$

We refer to [95, Ch. 5] for a detailed discussion of supersymmetric Dirac-type operators and the many explicit examples they represent.

Next, we strengthen the hypotheses on $\Phi$ to the effect that

$$\Phi \in C_b^2\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big), \quad d \in \mathbb{N}, \tag{1.9}$$

$$\Phi(x) = \Phi(x)^*, \quad x \in \mathbb{R}^n, \tag{1.10}$$

there exists $c > 0$, $R \geqslant 0$ such that

$$|\Phi(x)| \geqslant cI_d, \quad x \in \mathbb{R}^n \backslash B(0,R), \tag{1.11}$$

and there exists $\varepsilon > 1/2$ such that for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| < 3$, there is $\kappa > 0$ such that

$$\|(\partial^\alpha \Phi)(x)\| \leqslant \begin{cases} \kappa(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa(1+|x|)^{-1-\varepsilon}, & |\alpha| = 2, \end{cases} \quad x \in \mathbb{R}^n. \tag{1.12}$$

**Theorem 1.1.** *Let $n \in \mathbb{N}$ odd, $n \geqslant 3$. Under assumptions* (1.9)–(1.12) *on $\Phi$, the closed operator $L \coloneqq \mathcal{Q} + \Phi$ in $L^2(\mathbb{R}^n)^{2^{\hat{n}}d}$ is Fredholm with index given by the formula*

$$\begin{aligned} \operatorname{ind}(L) = \bigg(\frac{i}{8\pi}\bigg)^{(n-1)/2} \frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\dots i_n} \\ \times \int_{\Lambda S^{n-1}} \operatorname{tr}_{\mathbb{C}^d}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x))x_{i_n}\, d^{n-1}\sigma(x), \end{aligned} \tag{1.13}$$

*where*

$$U(x) := |\Phi(x)|^{-1}\Phi(x) = \operatorname{sgn}(\Phi(x)), \quad x \in \mathbb{R}^n.$$

Here $\varepsilon_{i_1\cdots i_n}$ denotes the totally anti-symmetric symbol in $n$ coordinates, $\operatorname{tr}_{\mathbb{C}^d}(\cdot)$ represents the matrix trace in $\mathbb{C}^{d\times d}$, $d^{n-1}\sigma(\cdot)$ is the surface measure on the unit sphere $S^{n-1}$ of $\mathbb{R}^n$, and we assumed $n \in \mathbb{N}$ to be odd since for algebraic reasons $L$ has vanishing Fredholm index in all even spatial dimensions $n$ (cf. (1.20) below).

Theorem 1.1 represents the principal result of this manuscript and under these hypotheses on $\Phi$ it is new as we suppose no additional asymptotic homogeneity properties on $\Phi$. In particular, it extends the original Callias formula for the index of $L$ to the hypotheses (1.9)–(1.12) on $\Phi$. We also note that at the end of this manuscript we take some first steps towards computing the Witten index of the operator $L$ under certain conditions on $\Phi$ in which $L$ ceases to be Fredholm, yet its Witten index is still given by a formula analogous to (1.13).

For the topological setting underlying the Callias index formula (1.13) we refer to the discussion by Bott and Seeley [14].

Next, we succinctly summarize the principal strategy of proof underlying formula (1.13). While at first we follow Callias' original strategy of proof, the bulk of our arguments necessarily differ sharply from those in [22] as some of the claims in [22] can clearly be disproved (see our subsequent discussion).

**Step (1): Computing Fredholm indices abstractly.** Let $\mathcal{H}$ be a separable Hilbert space, $m \in \mathbb{N}$, and $T \in \mathcal{B}(\mathcal{H}^m, \mathcal{H}^m)$. Define the *internal trace*, $\operatorname{tr}_m(T)$, *of* $T$ by

$$\operatorname{tr}_m(T) \coloneqq \sum_{j=1}^{m} T_{jj}. \tag{1.14}$$

Next, let $M$ be a densely defined, closed linear operator in $\mathcal{H}^m$, and introduce the abbreviation

$$B_M(z) := z\,\mathrm{tr}_m\big((M^*M+z)^{-1}-(MM^*+z)^{-1}\big), \quad z\in\varrho(-M^*M)\cap\varrho(-MM^*). \tag{1.15}$$

A basic result we employ to compute Fredholm indices then reads as follows:

**Theorem 1.2.** *Assume that $M$ is a densely defined, closed, and linear operator in $\mathcal{H}^m$, and suppose that $M$ is Fredholm. In addition, let $\{T_\Lambda\}_{\Lambda\in\mathbb{N}}$, $\{S_\Lambda^*\}_{\Lambda\in\mathbb{N}}$ be sequences in $\mathcal{B}(\mathcal{H})$, both strongly converging to $I_{\mathcal{H}}$ as $\Lambda\to\infty$, and introduce $S_\Lambda := S_\Lambda^{**}$, $\Lambda\in\mathbb{N}$. Assume that for each $\Lambda\in\mathbb{N}$, there exists $\delta_\Lambda>0$ with $\Omega_\Lambda := B(0,\delta_\Lambda)\backslash\{0\}\subseteq\varrho(-MM^*)\cap\varrho(-M^*M)$ and that the map*

$$\Omega_\Lambda\ni z\mapsto T_\Lambda B_M(z)S_\Lambda \tag{1.16}$$

*takes on values in $\mathcal{B}_1(\mathcal{H})$, such that*

$$\Omega_\Lambda\ni z\mapsto \mathrm{tr}_{\mathcal{H}}(|T_\Lambda B_M(z)S_\Lambda|)=\|T_\Lambda B_M(z)S_\Lambda\|_{\mathcal{B}_1(\mathcal{H})} \text{ is bounded (w.r.t. } z), \tag{1.17}$$

*where $\mathrm{tr}_{\mathcal{H}}(\cdot)$ represents the trace on $\mathcal{B}_1(\mathcal{H})$, the Schatten-von Neumann ideal of trace class operators on $\mathcal{H}$. Then,*

$$\mathrm{ind}(M)=\lim_{\Lambda\to\infty}\lim_{z\to 0}\mathrm{tr}_{\mathcal{H}}(T_\Lambda B_M(z)S_\Lambda). \tag{1.18}$$

*In addition, if $\delta := 2^{-1}\underline{\inf_{\Lambda\in\mathbb{N}}}(\delta_\Lambda)>0$ and $\Omega := B(0,\delta)\ni z\mapsto \mathrm{tr}_{\mathcal{H}}(T_\Lambda B_M(z)S_\Lambda)$ converges uniformly on $\overline{B(0,\delta)}$ to some function $F(\cdot)$ as $\Lambda\to\infty$. Then, one can interchange the limits $\Lambda\to\infty$ and $z\to 0$ in (1.18) and obtains,*

$$F(0)=\mathrm{ind}(M). \tag{1.19}$$

We emphasize that (1.18) and (1.19) represent a subtle, but crucial, deviation from the far simpler strategy employed in [22, Lemma 1] which entirely dispenses with the additional regularization factors $S_\Lambda$ and $T_\Lambda$, $\Lambda\in\mathbb{N}$. At this point we do not know if [22, Lemma 1] is valid, however, its proof is clearly invalid and we record a counterexample (kindly communicated to us by H. Vogt [98]) to the statement made on line 5 on p. 219 in the proof of [22, Lemma 1] later in Remark 3.5 $(i)$. After completing this project we became aware of an unpublished preprint by Arai [8] in which it was observed that the index regularization employed in [22] was insufficient.

**Step (2): Applying Step (1) to the operator $L$.** One now identifies $\mathcal{H}$ and $L^2(\mathbb{R}^n)$, $m$ and $2^{\widehat{n}}d$, $M$ and $L$, $T_\Lambda$ and the operator of multiplication by the characteristic function of the ball $B(0,\Lambda)\subset\mathbb{R}^n$ in $L^2(\mathbb{R}^n)$, denoted by $\chi_\Lambda$, and chooses $S_\Lambda^* = I_{L^2(\mathbb{R}^n)}$, $\Lambda\in\mathbb{N}$.

According to (1.18) and especially, (1.19), we are thus interested in computing the limit for $\Lambda\to\infty$ of $\mathrm{tr}(\chi_\Lambda B_L(z))$. Without loss of generality we restrict ourselves in the following to $n\in\mathbb{N}$ odd, as a detailed analysis shows that actually

$$B_L(z)=0 \text{ for } n\in\mathbb{N},\, n \text{ even}. \tag{1.20}$$

For $z\in\varrho(-LL^*)\cap\varrho(-L^*L)$ with $\mathrm{Re}(z)>-1$, and $n\in\mathbb{N}$ odd, $n\geqslant 3$, one then proceeds to prove that $\chi_\Lambda B_L(z)\in\mathcal{B}_1\big(L^2(\mathbb{R}^n)\big)$, and that the limit

$$f(z) := \lim_{\Lambda\to\infty}\mathrm{tr}_{L^2(\mathbb{R}^n)}(\chi_\Lambda B_L(z))$$

exists.

**Step (3): Explicitly compute** $f(z)$**.** A careful (and rather lengthy) evaluation of $f(z)$ yields

$$\begin{aligned} f(z) = (1+z)^{-n/2}\left(\frac{i}{8\pi}\right)^{(n-1)/2} \frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty}\frac{1}{2\Lambda}\sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\dots i_n} \\ \times \int_{\Lambda S^{n-1}} \operatorname{tr}_{\mathbb{C}^d}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x))x_{i_n}\, d^{n-1}\sigma(x), \\ z\in\varrho\left(-LL^*\right)\cap\varrho\left(-L^*L\right),\ \operatorname{Re}(z)>-1. \end{aligned} \tag{1.21}$$

However, at first we are only able to verify (1.21) for $\operatorname{Re}(z)$ sufficiently large (as a consequence of relying on Neumann series expansions for resolvents). In order to derive (1.21) also for $z$ in a neighborhood of 0, considerable additional efforts are required.

Indeed, for achieving the existence of the limit $\Lambda\to\infty$ in (1.21) for $z$ in a neighborhood of 0, we employ Montel's theorem and hence need to show that the family of analytic functions $\{z\mapsto \operatorname{tr}(\chi_\Lambda B_L(z))\}_\Lambda$ constitutes a locally bounded family, that is, one needs to show that for all compact $\Omega\subset\mathbb{C}_{\operatorname{Re}>-1}\cap\varrho(-L^*L)\cap\varrho(-LL^*)$,

$$\sup_{\Lambda>0}\sup_{z\in\Omega}|\operatorname{tr}(\chi_\Lambda B_L(z))|<\infty.$$

After proving local boundedness, we use Montel's theorem for deducing that at least for a sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ with $\Lambda_k \underset{k\to\infty}{\longrightarrow}\infty$, the limit $f:=\lim_{k\to\infty}\operatorname{tr}(\chi_{\Lambda_k}B_L(\cdot))$ exists in the compact open topology (i.e., the topology of uniform convergence on compacts). The explicit expression (1.21) for $f$ then follows by the principle of analytic continuation and so carries over to $z$ in a neighborhood of 0. In particular, since the limit $\lim_{\Lambda\to\infty}\operatorname{tr}(\chi_\Lambda B_L(0))$ exists and coincides with the index of $L$, we can then deduce that independently of the sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$, the limit $\lim_{\Lambda\to\infty}\operatorname{tr}(\chi_\Lambda B_L(\cdot))$ exists in the compact open topology and coincides with $f$ given in (1.21). Thus,

$$f(0)=\operatorname{ind}(L)$$

yields formula (1.13).

We also emphasize that in connection with Steps (1)–(3), we perform these calculations only in the special case of admissible or $\tau$-admissible potentials $\Phi$ (cf. Definitions 6.11 and 12.5) and then reduce the general case to $\tau$-admissible potentials.

It is clear from this short outline of our strategy of proof of Callias' index formula (1.13), that in the end, our proof requires a fair number of additional steps not present in [22].

Without entering any details at this point, we mention that one needs to distinguish the case $n=3$ from $n\geqslant 5$ as there are additional regularization steps necessary for $n=3$ due to the lack of regularity of certain integral kernels. In this context we mention that it is unclear to us how continuity of the integral kernel of $J_z^i$ on the diagonal, as claimed in [22, p. 224, line 6 from below], can be proved. Given our detailed approach, the number of resolvents applied is simply not large enough to conclude continuity (see, in particular, Section 7).

Perhaps, more drastically, trace class properties of certain integral operators are merely dealt with by checking integrability of the integral kernel on the diagonal, see, for instance, the proof of [22, Lemma 5, p. 225].

In addition, the claim that the expression

$$\sum_{i_1,\dots,i_n} \varepsilon_{i_1\dots i_n} \operatorname{tr}\left((\partial_{i_1}\Phi)(x)\dots(\partial_{i_n}\Phi)(x)\right) = 0, \quad x \in \mathbb{R}^n, \tag{1.22}$$

vanishes identically, is made on [22, p. 226]. A simple counter example can (locally) be constructed by demanding that $\Phi\colon \mathbb{R}^3 \to \mathbb{C}^{2\times 2}$ is bounded, $\Phi \in C^\infty\big(\mathbb{R}^3;\mathbb{C}^{2\times 2}\big)$, and such that for one particular $x_0 \in \mathbb{R}^3$,

$$(\partial_1\Phi)(x_0) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad (\partial_2\Phi)(x_0) = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad (\partial_1\Phi)(x_0) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

In this case one verifies that

$$\sum_{i_1,i_2,i_3} \varepsilon_{i_1 i_2 i_3} \operatorname{tr}\left((\partial_{i_1}\Phi)(x_0)(\partial_{i_2}\Phi)(x_0)(\partial_{i_3}\Phi)(x_0)\right) = 24i.$$

These shortcomings in the arguments presented in [22] not withstanding, Callias' formula (1.13) is remarkable for its simplicity, as has been pointed out before by various authors. In particular, it is simpler, yet consistent with the Fedosov–Hörmander formula [42], [43], [44], [45], [66], [67, Sect. 19.3] (derived with the help of the pseudo-differential operator calculus), as discussed, for instance, in [3], [14], [91]. More precisely, the Fedosov–Hörmander formula reads as follows,

$$\operatorname{ind}(L) = -\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \int_{\partial B} \operatorname{tr}\left(\left(\sigma_L^{-1}\mathrm{d}\sigma_L\right)^{\wedge(2n-1)}\right). \tag{1.23}$$

Here $\sigma_L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^{2^{\widehat{n}}d\times 2^{\widehat{n}}d}$ is the symbol of $L$ given by

$$\sigma_L(\xi,x) = \sum_{j=1}^n \gamma_{j,n} i\xi \otimes I_{2^{\widehat{n}}} + I_d \otimes \Phi(x), \quad \xi, x \in \mathbb{R}^n,$$

$B \subseteq \mathbb{R}^{2n}$ is a ball of sufficiently large radius centered at the origin such that $\sigma_L$ is invertible outside $B$, the orientation of $\mathbb{R}^n\times\mathbb{R}^n$ is given by $dx_1\wedge d\xi_1\wedge\cdots\wedge dx_n\wedge d\xi_n > 0$, and $\left(\sigma_L^{-1}\mathrm{d}\sigma_L\right)^{\wedge(2n-1)}$ is evaluated as a matrix product upon replacing ordinary multiplication by the exterior product.

The Callias index formula properly restated as the Fedosov–Hörmander formula and connections with *half-bounded states* were also discussed in [30]. Moreover, with the help of the Cordes–Illner theory (see [36, 68] and the references in [85]), [85] established that the Fedosov–Hörmander formula can also be used for computing the index, if $L$ is considered as an operator from the Sobolev space $W^{1,p}(\mathbb{R}^n)^{2^{\widehat{n}}d}$ to $L^p(\mathbb{R}^n)^{2^{\widehat{n}}d}$ for some $p \in (1,\infty)$. In addition, [86] (see also [87]) established the validity of the Fedosov–Hörmander formula assuming the low regularity $\Phi \in C^1$ only (plus vanishing of derivatives at infinity).

Callias employed Witten's resolvent regularization inherent in (1.15), (1.18), (1.19), and we followed this device in this manuscript. For extensions to higher powers of resolvents we refer to [94]. For connections between supersymmetric quantum mechanics, scattering theory and their connections with Witten's resolvent regularized index for Dirac-type operators in various space dimensions, and matrix-valued (resp., operator-valued) coeffcients, we refer, for instance, to [6], [7], [12], [13], [20], [23], [24], [31], [75], [76], [79, Chs. IX, X], [80].

The index problem for Dirac operators defined on complete Riemanniann manifolds has also been studied in [58] on the basis of relative index theorems (see also [88]). Based on this approach, [4] found a generalized version of the Callias

index formula, which was further developed and connected with the Atiyah–Singer index theorem in [5] (see also [28], [29], [41], [60] in this context). Independently, [89] found an alternative proof for the main result in [5], reducing the index problem for the Dirac operator on a non-compact manifold to the compact case, thus making the index theorem in [10] applicable. Generalizing results in [4], and also using the Atiyah–Singer index theorem, [18] (see also [17]) derive index formulas on manifolds, containing the Callias index formula as special case.

For further generalizations of the index theorem for the Dirac operator to particular manifolds, we refer to [48]. In addition, certain classes of Dirac operators on even-dimensional manifolds are studied in [49], [47], [50], [51] employing $K$ or $KK$-theory. The utility of $KK$-theory in view of the Callias index formula can also be seen in [74], where a short proof for the main results in [4] is given. Additional connections between $K$-theory and index theory for Dirac-type operators have been established, for instance, in [21], [32], [33], [72], [73]. A rather different direction of index theory employing cyclic homology, aimed at even dimensional Dirac-type operators which generally are non-Fredholm, was undertaken in [25] (see also [26]).

The approach to calculating Fredholm indices initiated by Callias [22] also had a profound influence on theoretical physics as is amply demonstrated by the following references [15], [34], [40], [46], [61], [62], [63], [64], [65], [69], [82], [83], [101], [102], [103], [104], and the literature cited therein.

Returning to Theorem 1.1, we emphasize again that our derivation of the Callias index formula (1.13) under conditions (1.9)–(1.12) on $\Phi$ is new as the references just mentioned either do not derive an explicit formula for $\operatorname{ind}(L)$ in terms of $\Phi$, or else, derive the Fedosov–Hörmander formula for $\operatorname{ind}(L)$. All previous derivations of (1.13) made some assumptions on $\Phi$ to the effect that asymptotically, $\Phi$ had to be homogeneous of degree zero. We entirely dispensed with this condition in this manuscript.

We conclude this introduction with a brief description of the contents of each section. Our notational conventions are summarized in Section 2. Section 3 is devoted to computing Fredholm indices employing Witten's resolvent regularization. Schatten–von Neumann classes and trace class estimates are treated in Section 4. Pointwise bounds for integral kernels are developed in Section 5. The operator $L$ underlying this manuscript is presented in Section 6. Trace class results, fundamental for deriving formula (1.21) for $f(z)$, are established in Section 7; estimates for integral kernels on the diagonal and the computation of the trace of $\chi_\Lambda B_L(z)$ are discussed in Section 8; the special case $n = 3$ is treated in Section 9. In Section 10 we formulate our principal result, Theorem 10.2 (equivalently, Theorem 1.1) and discuss some consequences of formula (1.13). Perturbation theory of Helmholtz resolvents (Green's functions) is isolated in Section 11. The proof of Theorem 10.2 for smooth $\Phi$ is presented in Section 12; the case of general $\Phi$ satisfying (1.9)–(1.12) is concluded in Section 13. The final Section 14 is devoted to a particular class of non-Fredholm operators $L$ and the associated Witten index. Appendix A presents a concise construction of the Euclidean Dirac algebra, and Appendix B constructs an explicit counterexample to the trace class assertion in [22, Lemma 5].

**Acknowledgments.** We thank Bernhelm Booß-Bavnbek, Alan Carey, Stamatis Dostoglou, Yuri Latushkin, Marius Mitrea, and Andreas Wipf for valuable discussions and/or correspondence. We are especially grateful to Alan Carey for extensive correspondence and hints to the literature, and to Stamatis Dostoglou for a critical

reading of a substantial part of our manuscript and for many constructive comments that helped improving our exposition. We also thank Ralph Chill, Martin Körber, Rainer Picard, and Sascha Trostorff for useful discussions, particularly, concerning the counterexample discussed in Appendix B. In addition, we are indebted to Hendrik Vogt for kindly communicating the example in Remark 3.5.

M.W. gratefully acknowledges the hospitality of the Department of Mathematics at the University of Missouri, USA, extended to him during two one-month visits in the spring of 2014 and 2015. Moreover, he is particularly indebted to Yuri Tomilov in connection with the support received from the EU grant "AOS", FP7-PEOPLE-2012-IRSES, No. 318910.

## 2. Notational Conventions

For convenience of the reader we now summarize most of our notational conventions used throughout this manuscript.

We find it convenient to employ the abbreviations, $\mathbb{N}_{\geqslant k} := \mathbb{N} \cap [k, \infty)$, $k \in \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}_{\mathrm{Re}>a} \coloneqq \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > a\}$, $a \in \mathbb{R}$.

The identity matrix in $\mathbb{C}^r$ will be denoted by $I_r$, $r \in \mathbb{N}$.

Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot\,,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

Next, let $T$ be a linear operator mapping (a subspace of) a Banach space into another, with $\mathrm{dom}(T)$, $\ker(T)$, and $\mathrm{ran}(T)$ denoting the domain, kernel (i.e., null space), and range of $T$. The spectrum and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$ and $\varrho(\cdot)$. For resolvents of closed operators $T$ acting on $\mathrm{dom}(T) \subseteq \mathcal{H}$, we will frequently write $(T-z)^{-1}$ rather than the precise $(T-zI_{\mathcal{H}})^{-1}$, $z \in \varrho(T)$.

The Banach spaces of bounded and compact linear operators in $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. The Schatten–von Neumann ideals of compact linear operators on $\mathcal{H}$ corresponding to $\ell^p$-summable singular values will be denoted by $\mathcal{B}_p(\mathcal{H})$ or, if the Hilbert space under consideration is clear from the context (and, especially, for brevity in connection with proofs) just by $\mathcal{B}_p$, $p \in [1, \infty)$. The norms on the respective spaces will be noted by $\|T\|_{\mathcal{B}_p(\mathcal{H})}$ for $T \in \mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$, and for ease of notation we will occasionally identify $\|T\|_{\mathcal{B}(\mathcal{H})}$ with $\|T\|_{\mathcal{B}_\infty(\mathcal{H})}$ for $T \in \mathcal{B}(\mathcal{H})$, but caution the reader that it is the set of compact operators on $\mathcal{H}$ that is denoted by $\mathcal{B}_\infty(\mathcal{H})$. Similarly, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ will be used for bounded and compact operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$. Throughout this manuscript, if $\mathcal{X}$ denotes a Banach space, $\mathcal{X}^*$ denotes the *adjoint space* of continuous conjugate linear functionals on $\mathcal{X}$, that is, the *conjugate dual space* of $\mathcal{X}$ (rather than the usual dual space of continuous linear functionals on $\mathcal{X}$). This avoids the well-known awkward distinction between adjoint operators in Banach and Hilbert spaces (cf., e.g., the pertinent discussion in [39, p. 3–4]). In connection with bounded linear functionals on $\mathcal{X}$ we will employ the usual bracket notation $\langle\cdot\,,\cdot\rangle_{\mathcal{X}^*,\mathcal{X}}$ for pairings between elements of $\mathcal{X}^*$ and $\mathcal{X}$.

Whenever estimating the operator norm or a particular trace ideal norm of a finite product of operators, $A_1 A_2 \cdots A_N$, with $A_j \in \mathcal{B}(\mathcal{H})$, $j \in \{1, \dots, N\}$, $N \in \mathbb{N}$, we will simplify notation and write

$$\prod_{j=1}^{N} A_j, \tag{2.1}$$

disregarding any noncommutativity issues of the operators $A_j$, $j \in \{1, \dots, N\}$. This is of course permitted due to standard ideal properties and the associated (noncommutative) Hölder-type inequalities (see, e.g., [55, Sect. III.7], [92, Ch. 2]). The same convention will be applied if operators mapping between several Hilbert spaces are involved.

We use the *commutator* symbol

$$[A, B] \coloneqq AB - BA \tag{2.2}$$

for suitable operators $A, B$. For unbounded $A$ and $B$ the natural domain of $[A, B]$ is the intersection of the respective domains of $AB$ and $BA$. In particular, $[A, B]$

is not closed in general. However, in the situations we are confronted with, we shall always be in the situation that $[A, B]$ is densely defined and bounded, in particular, it is closable with bounded closure. As this is always the case, we shall – in order to reduce a clumsy notation as much as possible – typically omit the closure bar (i.e., we use $[A, B]$ rather than $\overline{[A, B]}$). In fact, most of the operators under consideration can be extended to suitable distribution spaces, such that seemingly formal computations can be justified in the appropriate distribution space.

w-lim and s-lim denote weak and strong limits in $\mathcal{H}$ as well as limits in the weak and strong operator topology for operators in $\mathcal{B}(\mathcal{H})$, n-lim denotes the norm limit of bounded operators operators in $\mathcal{H}$ (i.e., in the topology of $\mathcal{B}(\mathcal{H})$).

$C_0^\infty(\mathbb{R}^n)$ denotes the space of infinitely often differentiable functions with compact support in $\mathbb{R}^n$. We typically suppress the Lebesgue meausure in $L^p$-spaces, $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n; d^n x)$, $\|\cdot\|_{L^p(\mathbb{R}^n; d^n x)} := \|\cdot\|_p$, and similarly, $L^p(\Omega) := L^p(\Omega; d^n x)$, $\Omega \subseteq \mathbb{R}^n$, $p \in [1, \infty) \cup \{\infty\}$. To avoid too lengthy expressions, we will frequently just write $I$ rather than the precise $I_{L^2(\mathbb{R}^n)}$, etc.

Sometimes we use the symbol $\langle \cdot\,, \cdot \rangle_{L^2(\mathbb{R}^n)}$ (or, for brevity, especially in proofs, simply $\langle \cdot\,, \cdot \rangle$), to indicate the fact that the scalar product $(\cdot\,, \cdot)_{L^2(\mathbb{R}^n)}$ in $L^2(\mathbb{R}^n)$ has been continuously extended to the pairing on the entire Sobolev scale, that is, we abreviate $\langle \cdot\,, \cdot \rangle_{L^2(\mathbb{R}^n)} := \langle \cdot\,, \cdot \rangle_{H^{-s}(\mathbb{R}^n), H^s(\mathbb{R}^n)}$, $s \geqslant 0$.

The unit sphere in $\mathbb{R}^n$ is denoted by $S^{n-1} := \{x \in \mathbb{R}^n \,|\, |x| = 1\}$, with $d^{n-1}\sigma(\cdot)$ representing the surface measure on $S^{n-1}$, $n \in \mathbb{N}_{\geqslant 2}$. The open ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ of radius $r_0 > 0$ is denoted by $B(x_0, r_0)$.

Since various matrix structures and tensor products are naturally associated with the Dirac-type operators studied in this manuscript, we had to simplify the notation in several respects to avoid entirely unmanagably long expressions. For example, given $d, \widehat{n} \in \mathbb{N}$, spaces such as $L^2(\mathbb{R}^n) \otimes \mathbb{C}^d$, $L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^{\widehat{n}}}$, and $L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^{\widehat{n}}} \otimes \mathbb{C}^d$ (and analogously for Sobolev spaces) will simply be denoted by $L^2(\mathbb{R}^n)^d$, $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$, and $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$, respectively.

In addition, given a $d \times d$ matrix $\Phi \colon \mathbb{R}^n \to \mathbb{C}^{d \times d}$ with entries given by bounded measurable functions, and given an element $\psi \otimes \phi \in L^2(\mathbb{R}^n)^{2^{\widehat{n}}} \otimes \mathbb{C}^d$, we will frequently adhere to a slight abuse of notation and employ the symbol $\Phi$ also in the context of the operation

$$\Phi \colon \psi \otimes \phi \mapsto (x \mapsto \psi(x) \otimes \Phi(x)\phi)\,, \tag{2.3}$$

and acordingly then shorten this even further to

$$\Phi \colon \psi\,\phi \mapsto (x \mapsto \psi(x)\Phi(x)\phi)\,, \tag{2.4}$$

Moreover, in connection with constant, invertible $m \times m$ matrices $\alpha \in \mathbb{C}^{m \times m}$ and scalar differential expressions such as $\partial_j$, $\Delta$, etc., we will use the notation

$$\alpha\partial_j = \partial_j\alpha, \quad \alpha\Delta = \Delta\alpha \tag{2.5}$$

(with equality of domains) when applying these differential expressions to sufficiently regular functions of the type $\eta(\cdot) \otimes c$, $c \in \mathbb{C}^m$, abbreviated again by $\eta(\cdot)\,c$.

In the context of matrix-valued operators we also agree to use the following notational conventions: Given a scalar function $f$ on $\mathbb{R}^n$, or a scalar linear operator $R$ in $L^2(\mathbb{R}^n)$, we will frequently identify $f$ or $R$ with the diagonal matrices $f\,I_m$ or $R\,I_m$ in $L^2(\mathbb{R}^n)^{m \times m}$ for appropriate $m \in \mathbb{N}$.

*Remark* 2.1. We will identify a function $\Phi$ with its corresponding multiplication operator of multiplying by this function in a suitable function space. In doing so, for a differential operator $\mathcal{Q}$, we will distinguish between the expression $\mathcal{Q}\Phi$ and $(\mathcal{Q}\Phi)$ and, similarly, for other differential operators. Namely, $\mathcal{Q}\Phi$ denotes the *composition* of the two operators $\mathcal{Q}$ and $\Phi$, whereas $(\mathcal{Q}\Phi)$ denotes the *multiplication operator* of multplying by the function $x \mapsto (\mathcal{Q}\Phi)(x)$. $\diamond$

## 3. Functional Analytic Preliminaries

In this section we shall summarize the results obtained by Callias in [22, Lemmas 1 and 2]. We emphasize that we only succeeded to prove [22, Lemma 1] under the stronger condition that the trace norm of the operator under consideration is bounded on a punctured neighborhood around the origin. To begin, we recall the setting of [22, Section II, p. 218]:

**Definition 3.1.** *Let $\mathcal{H}$ be a separable Hilbert space, $m \in \mathbb{N}$, and $T \in \mathcal{B}(\mathcal{H}^m, \mathcal{H}^m)$, a bounded linear operator from $\mathcal{H}^m$ to $\mathcal{H}^m$. Denoting by $\iota_j \colon \mathcal{H} \to \mathcal{H}^m$ the canonical embedding defined by $\iota_j h := \{\delta_{kj} h\}_{k\in\{1,\dots,m\}}$, we introduce $T_{jk} := \iota_j^* T \iota_k$ for $j, k \in \{1, \dots, m\}$. We define the* internal trace, $\operatorname{tr}_m(T)$, of $T$ *being the linear operator on $\mathcal{H}$ given by*

$$\operatorname{tr}_m(T) := \sum_{j=1}^m T_{jj}. \tag{3.1}$$

*Next, let $M$ be a densely defined, closed linear operator in $\mathcal{H}^m$ and introduce Witten's resolvent regularization via*

$$\begin{aligned} B_M(z) := z \operatorname{tr}_m \big( (M^*M + z)^{-1} - (MM^* + z)^{-1} \big) \in \mathcal{B}(\mathcal{H}), \\ z \in \varrho(-M^*M) \cap \varrho(-MM^*). \end{aligned} \tag{3.2}$$

*We will denote by $\operatorname{tr}_{\mathcal{H}}(\cdot)$ the trace on $\mathcal{B}_1(\mathcal{H})$, the Schatten-von Neumann ideal of trace class operators on $\mathcal{H}$.*

*Remark* 3.2. Let $\mathcal{H}$ be a Hilbert space and let $m \in \mathbb{N}$, $T \in \mathcal{B}_1(\mathcal{H}^m)$ and let $T_{jk} = \iota_j^* T \iota_k$, $j, k \in \{1, \dots, m\}$ as in Definition 3.1. Then boundedness of $\iota_j^*, \iota_k$, $j, k \in \{1, \dots, m\}$, and exploiting the ideal property of $\mathcal{B}_1(\mathcal{H}^m)$ yields

$$T_{jk} \in \mathcal{B}_1(\mathcal{H}), \quad j, k \in \{1, \dots, m\},$$

in particular,

$$\operatorname{tr}_m(T) \in \mathcal{B}_1(\mathcal{H}).$$

$\diamond$

It should be noted that in general, the internal trace does *not* satisfy the cyclicity property in the sense that for $A, B \in \mathcal{B}(\mathcal{H}^m, \mathcal{H}^m)$,

$$\operatorname{tr}_m(AB) \neq \operatorname{tr}_m(BA).$$

However, if one of the operators is actually a matrix with entries in $\mathbb{C}$, then such a result holds:

**Proposition 3.3.** *Let $\mathcal{H}$ be a Hilbert space, $m \in \mathbb{N}$, $A \in \mathcal{B}(\mathcal{H}^m, \mathcal{H}^m)$, $B \in \mathbb{C}^{m\times m}$. Then*

$$\operatorname{tr}_m(AB) = \operatorname{tr}_m(BA).$$

*Proof.* We have $A = (A_{ij})_{i,j\in\{1,\dots,m\}}$ and $B = (B_{ij})_{i,j\in\{1,\dots,m\}}$ with $A_{ij} \in \mathcal{B}(\mathcal{H})$ as in Definition 3.1 and $B_{ij} \in \mathbb{C}$. Then

$$AB = \Bigg( \sum_{k\in\{1,\dots,m\}} A_{ik} B_{kj} \Bigg)_{i,j\in\{1,\dots,m\}} \text{ and } BA = \Bigg( \sum_{k\in\{1,\dots,m\}} B_{ik} A_{kj} \Bigg)_{i,j\in\{1,\dots,m\}}.$$

Hence,

$$\operatorname{tr}_m(AB) = \sum_{j=1}^m \sum_{k=1}^m A_{jk} B_{kj}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \sum_{k=1}^{m} B_{kj} A_{jk} \\
&= \sum_{k=1}^{m} \sum_{j=1}^{m} \, B_{kj} A_{jk} = \operatorname{tr}_m(BA). \qquad \square
\end{aligned}
$$

Next, we need a result of the type of [22, Lemma 1], in fact, we need an additional generalization of [22, Lemma 1] in order to be able to apply it to our situation. We shall briefly recall the notions used in the next result: Given a Hilbert space $\mathcal{K}$, a *Fredholm operator* $S\colon \operatorname{dom}(S) \subseteq \mathcal{K} \to \mathcal{K}$, denoted by $S \in \Phi(\mathcal{K})$, is defined by $S$ being a densely defined, closed, linear operator with finite-dimensional nullspace, $\dim(\ker(S)) < \infty$, and closed range, $\operatorname{ran}(S)$, being finite-codimensional, $\dim(\ker(S^*)) < \infty$. The *Fredholm index*, $\operatorname{ind}(S)$, of a Fredholm operator $S$ is then the difference of the dimension of the nullspace and codimension of the range, that is,

$$
\operatorname{ind}(S) = \dim(\ker(S)) - \dim(\ker(S^*)).
$$

Basic facts on Fredholm operators will be recalled at the end of this section. For the next lemma, we shall also use the notion of convergence in the strong operator topology, that is, a sequence $\{T_\Lambda\}_{\Lambda\in\mathbb{N}}$ of bounded linear operators in a Hilbert space $\mathcal{H}$ is said to converge to some $T_\infty \in \mathcal{B}(\mathcal{H})$ in the *strong operator topology*, $\operatorname{s-lim}_{\Lambda\to\infty} T_\Lambda = T_\infty$, if for all $\phi \in \mathcal{H}$, we have

$$
\lim_{\Lambda\to\infty} T_\Lambda \phi = T_\infty \phi.
$$

Our (generalized) version of [22, Lemma 1] then reads as follows.

**Theorem 3.4.** *In the situation of Definition* 3.1 *assume that $M$ is Fredholm, and that $\{T_\Lambda\}_{\Lambda\in\mathbb{N}}$, $\{S_\Lambda^*\}_{\Lambda\in\mathbb{N}}$ are sequences in $\mathcal{B}(\mathcal{H})$, both converging to $I_{\mathcal{H}}$ in the strong operator topology as $\lambda \to \infty$, and introduce $S_\Lambda := S_\Lambda^{**}$, $\Lambda \in \mathbb{N}$. Let $B_M(\cdot)$ be given by* (3.2),

$$
B_M(z) := z \operatorname{tr}_m \big((M^*M + z)^{-1} - (MM^* + z)^{-1}\big), \quad z \in \varrho(-M^*M) \cap \varrho(-MM^*). \tag{3.3}
$$

*Assume that for each $\Lambda \in \mathbb{N}$, there exists $\delta_\Lambda > 0$ with $\Omega_\Lambda := B(0,\delta_\Lambda)\backslash\{0\} \subseteq \varrho(-MM^*) \cap \varrho(-M^*M)$ and that the map*

$$
\Omega_\Lambda \ni z \mapsto T_\Lambda B_M(z) S_\Lambda
$$

*takes on values in $\mathcal{B}_1(\mathcal{H})$, such that*

$$
\Omega_\Lambda \ni z \mapsto \operatorname{tr}_{\mathcal{H}}(|T_\Lambda B_M(z) S_\Lambda|) = \|T_\Lambda B_M(z) S_\Lambda\|_{\mathcal{B}_1(\mathcal{H})}
$$

*is bounded with respect to $z$. Then*

$$
\operatorname{ind}(M) = \lim_{\Lambda\to\infty} \lim_{z\to 0} \operatorname{tr}_{\mathcal{H}}(T_\Lambda B_M(z) S_\Lambda). \tag{3.4}
$$

*In addition, if $\delta := \frac{1}{2}\inf_{\Lambda\in\mathbb{N}}(\delta_\Lambda) > 0$ and $\Omega := B(0,\delta) \ni z \mapsto \operatorname{tr}_{\mathcal{H}}(T_\Lambda B_M(z) S_\Lambda)$ converges uniformly on $\overline{B(0,\delta)}$ to some function $F(\cdot)$ as $\Lambda \to \infty$. Then, one can interchange the limits $\Lambda \to \infty$ and $z \to 0$ in* (3.4) *and obtains,*

$$
F(0) = \operatorname{ind}(M). \tag{3.5}
$$

*Proof.* By the Fredholm property of $M$, one deduces that $M^*M$ and $MM^*$ are Fredholm and, if 0 lies in the spectrum of either $M^*M$ or $MM^*$ it is an isolated eigenvalue of finite multiplicity. As $M$ is Fredholm, $\operatorname{ran}(M)$ is closed. Hence, $\operatorname{ran}(M) = \ker(M^*)^{\perp}$, and since $M$ is closed, $\ker(M^*M) = \ker(M)$, as well as, $\ker(MM^*) = \ker(M^*)$. Denote by $P_\pm \colon \mathcal{H}^m \to \mathcal{H}^m$ the orthogonal projection onto $\ker(M)$ and $\ker(M^*)$, respectively. Since by hypothesis $P_\pm$ are finite-dimensional operators, so is $\operatorname{tr}_m(P_\pm)$. Moreover, we have

$$\operatorname{tr}_{\mathcal{H}}(\operatorname{tr}_m(P_\pm)) = \operatorname{tr}_{\mathcal{H}^m}(P_\pm) = \dim(\operatorname{ran}(P_\pm)).$$

Indeed, the last equality being clear, we only need to show the first one. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Let $\iota_j \colon \mathcal{H} \to \mathcal{H}^m$ be the canonical embedding given by $\iota_j h \coloneqq \{\delta_{\ell j} h\}_{\ell\in\{1,\ldots,m\}}$ for all $j \in \{1,\ldots,m\}$. Then it is clear that $\{\iota_j\phi_k\}_{j\in\{1,\ldots,m\},k\in\mathbb{N}}$ constitutes an orthonormal basis for $\mathcal{H}^m$. We have

$$\begin{aligned}
\operatorname{tr}_{\mathcal{H}^m}(P_\pm) &= \sum_{j=1}^m \sum_{k\in\mathbb{N}} (\iota_j\phi_k, P_\pm\iota_j\phi_k)_{\mathcal{H}^m} = \sum_{j=1}^m \sum_{k\in\mathbb{N}} (\phi_k, \iota_j^* P_\pm \iota_j\phi_k)_{\mathcal{H}} \\
&= \sum_{k\in\mathbb{N}} \sum_{j=1}^m (\phi_k, P_{\pm,jj}\phi_k)_{\mathcal{H}} = \sum_{k\in\mathbb{N}} \left(\phi_k, \sum_{j=1}^m P_{\pm,jj}\phi_k\right)_{\mathcal{H}} \\
&= \sum_{k\in\mathbb{N}} (\phi_k, \operatorname{tr}_m(P_\pm)\phi_k)_{\mathcal{H}} = \operatorname{tr}_{\mathcal{H}}(\operatorname{tr}_m(P_\pm)).
\end{aligned}$$

Next, define for $\Lambda \in \mathbb{N}$,

$$\begin{aligned}
\Omega_\Lambda \ni z \mapsto \widetilde{B}_\Lambda(z) &\coloneqq T_\Lambda \big[\operatorname{tr}_m\left(z(M^*M+z)^{-1}\right) - \operatorname{tr}_m(P_+) - \operatorname{tr}_m\left(z(MM^*+z)^{-1}\right) \\
&\qquad + \operatorname{tr}_m(P_-)\big] S_\Lambda \\
&= T_\Lambda B_M(z) S_\Lambda - T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda + T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda.
\end{aligned}$$

By [71, Sect. III.6.5],

$$z(M^*M+z)^{-1} - P_+ \underset{z\to 0}{\longrightarrow} 0$$

in operator norm, and similarly for $z \mapsto z(MM^*+z)^{-1} - P_-$. We note that $\widetilde{B}_\Lambda(z) \in \mathcal{B}_1(\mathcal{H})$, $z \in \Omega_\Lambda$. Since $\Omega_\Lambda \ni z \mapsto \operatorname{tr}_{\mathcal{H}}(|T_\Lambda B_M(z) S_\Lambda|)$ is bounded, so is $\Omega_\Lambda \ni z \mapsto \operatorname{tr}_{\mathcal{H}}\left(\big|\widetilde{B}_\Lambda(z)\big|\right)$. For the boundedness of $\Omega_\Lambda \ni z \mapsto \operatorname{tr}_{\mathcal{H}}\left(\big|\widetilde{B}_\Lambda(z)\big|\right)$ it suffices to observe that

$$\begin{aligned}
\operatorname{tr}_{\mathcal{H}}\left(\big|\widetilde{B}_\Lambda(z)\big|\right) &= \operatorname{tr}_{\mathcal{H}}\left(\big|T_\Lambda B_M(z) S_\Lambda - T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda + T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\big|\right) \\
&= \big\|T_\Lambda B_M(z) S_\Lambda - T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda + T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\big\|_{\mathcal{B}_1(\mathcal{H})} \\
&\leqslant \big\|T_\Lambda B_M(z) S_\Lambda\big\|_{\mathcal{B}_1(\mathcal{H})} \\
&\quad + \big\|T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda\big\|_{\mathcal{B}_1(\mathcal{H})} + \big\|T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\big\|_{\mathcal{B}_1(\mathcal{H})} \\
&= \operatorname{tr}_{\mathcal{H}}\left(\big|T_\Lambda B_M(z) S_\Lambda\big|\right) \\
&\quad + \operatorname{tr}_{\mathcal{H}}\left(\big|T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda\big|\right) + \operatorname{tr}_{\mathcal{H}}\left(\big|T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\big|\right), \quad z \in \Omega_\Lambda,
\end{aligned}$$

and using that the last two summands correspond to traces of finite-rank operators. Thus, from the analyticity of $\operatorname{tr}_{\mathcal{H}}\left(\widetilde{B}_\Lambda(\cdot)F\right)$ for every finite-rank operator $F$ on $\mathcal{H}$, one deduces that $\widetilde{B}_\Lambda(\cdot)$ is analytic in the $\mathcal{B}_1(\mathcal{H})$-norm, see, for instance, [9, Proposition A.3] (or [100, Theorem A.4.3]). More precisely, in [100, Theorem A.4.3] there is the following characterization of analyticity of Banach space valued functions: A function $h \colon \mathcal{U} \to \mathcal{X}$ for some open $\mathcal{U} \subseteq \mathbb{C}$ and some Banach space $\mathcal{X}$

is analytic if and only if $\mathcal{U} \ni z \mapsto \|h(z)\|_{\mathcal{X}}$ is bounded on compact subsets of $\mathcal{U}$ and $z \mapsto \langle h(z), x' \rangle$ is analytic for all $x' \in V$ with $V \subseteq \mathcal{X}'$ being a norming set for $\mathcal{X}$. Thus, it suffices to apply [100, Theorem A.4.3] to $\mathcal{X} = \mathcal{B}_1(\mathcal{H})$ as underlying Banach space, and to observe that the space of finite-rank operators forms a norming subset of $\mathcal{B}_1(\mathcal{H})$ (cf. [9, Proposition A.3].

By Riemann's theorem on removable singularities, one deduces that $\widetilde{B}_\Lambda(\cdot)$ is analytic at 0 with respect to the $\mathcal{B}_1(\mathcal{H})$-norm. As $\widetilde{B}_\Lambda(\cdot)$ is also norm analytic in $\mathcal{B}(\mathcal{H})$, and tends to 0 as $z \to 0$, one gets that $\widetilde{B}_\Lambda(z)$ tends to 0 as $z \to 0$ in $\mathcal{B}_1(\mathcal{H})$-norm. Hence,

$$\begin{aligned}
\lim_{z\to 0} \operatorname{tr}_{\mathcal{H}}(T_\Lambda B_M(z) S_\Lambda) &= \lim_{z\to 0} \Big( \operatorname{tr}_{\mathcal{H}} \big(\widetilde{B}_\Lambda(z)\big) + \operatorname{tr}_{\mathcal{H}} \left(T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda\right) \\
&\qquad - \operatorname{tr}_{\mathcal{H}} \left(T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\right) \Big) \\
&= 0 + \operatorname{tr}_{\mathcal{H}} \left(T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda\right) - \operatorname{tr}_{\mathcal{H}} \left(T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\right).
\end{aligned}$$

Since $\text{s-lim}_{\Lambda\to\infty} T_\Lambda, S_\Lambda^* = I_{\mathcal{H}}$, one obtains $T_\Lambda \operatorname{tr}_m(P_\pm) S_\Lambda \underset{\Lambda\to\infty}{\longrightarrow} \operatorname{tr}_m(P_\pm)$ in $\mathcal{B}_1(\mathcal{H})$ (see, e.g., [105, Lemma 6.1.3]). Thus,

$$\lim_{\Lambda\to\infty} \operatorname{tr}_{\mathcal{H}}(T_\Lambda \operatorname{tr}_m(P_\pm) S_\Lambda) = \operatorname{tr}_{\mathcal{H}}(\operatorname{tr}_m(P_\pm)) = \operatorname{tr}_{\mathcal{H}}(P_\pm).$$

Hence,

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \lim_{z\to 0} \operatorname{tr}_{\mathcal{H}}(T_\Lambda B_M(z) S_\Lambda) \\
&\quad = 0 + \lim_{\Lambda\to\infty} \Big( \operatorname{tr}_{\mathcal{H}} \left(T_\Lambda \operatorname{tr}_m(P_+) S_\Lambda\right) - \operatorname{tr}_{\mathcal{H}} \left(T_\Lambda \operatorname{tr}_m(P_-) S_\Lambda\right) \Big) \\
&\quad = \dim(\operatorname{ran}(P_+)) - \dim(\operatorname{ran}(P_-)) \\
&\quad = \dim(\ker(M)) - \dim(\ker(M^*)) \\
&\quad = \operatorname{ind}(M).
\end{aligned}$$

Finally, for the purpose of proving the last statement of the theorem, define $F_\Lambda \colon \Omega \ni z \mapsto \operatorname{tr}_{\mathcal{H}}(T_\Lambda B_M(z) S_\Lambda)$. Since $\{F_\Lambda\}_\Lambda$ converges uniformly to $F$, one infers that $F$ is continuous on $\Omega$. Thus,

$$\operatorname{ind}(M) = \lim_{\Lambda\to\infty} \lim_{z\to 0} F_\Lambda(z) = \lim_{\Lambda\to\infty} F_\Lambda(0) = F(0) = \lim_{z\to 0} F(z) = \lim_{z\to 0} \lim_{\Lambda\to\infty} F_\Lambda(z). \qquad \square$$

In connection with the last part of Theorem 3.4 we note that the (limit of the) map $z \mapsto F(z)$ in 0 may be regarded as a *generalized* Witten index (see, e.g., [12], [53] and the references therein, as well as Section 14).

*Remark* 3.5. (*i*) While [22, p. 218, Lemma 1] might be valid as stated, it remains unclear, how the assertion that is stated in line 5 on page 219 comes about. The author infers the following: Let $\mathcal{H}$ be a separable Hilbert space, $\Gamma \subseteq \mathbb{C}$ open with $0 \in \partial\Gamma$, $B \colon \Gamma \to \mathcal{B}(\mathcal{H})$ analytic. Assume that $B(z) \in \mathcal{B}_1(\mathcal{H})$ for all $z \in \Gamma$, $\Gamma \ni z \mapsto \|B(z)\|_{\mathcal{B}_1(\mathcal{H})}$ is bounded and $\|B(z)\|_{\mathcal{B}(\mathcal{H})} \to 0$ as $z \to 0$. Then for an orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$,

$$\operatorname{tr}_{\mathcal{H}}(B(z)) = \sum_{k=1}^{\infty} (\phi_k, B(z)\phi_k)_{\mathcal{H}} \underset{z\to 0}{\longrightarrow} 0. \tag{3.6}$$

This statement is invalid as the following example, kindly communicated to us by H. Vogt [98], shows: For the orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ define the family of operators

$B(\cdot)$ by

$$B(z)\phi_k := ze^{-(k-1)z}\phi_k, \quad z \in \Gamma := \{z \in \mathbb{C} \,|\, |\arg(z)| < (\pi/4),\, |z| < 1\}, \quad k \in \mathbb{N}.$$

Then

$$\begin{aligned}\|B(z)\|_{\mathcal{B}_1(\mathcal{H})} &= \operatorname{tr}_{\mathcal{H}}(|B(z)|) = \sum_{k=1}^{\infty}(\phi_k, |B(z)|\, \phi_k)_{\mathcal{H}} = \sum_{k=1}^{\infty} |z|\, e^{-(k-1)\operatorname{Re}(z)} \\ &= |z| \sum_{k=0}^{\infty} \Big(e^{-\operatorname{Re}(z)}\Big)^k = \frac{|z|}{1-e^{-\operatorname{Re}(z)}}\end{aligned}$$

remains bounded for $z \in \Gamma$. Moreover, $\text{n-lim}_{z\to 0} B(z) = 0$ in $\mathcal{B}(\mathcal{H})$. However,

$$\operatorname{tr}_{\mathcal{H}}(B(z)) = \sum_{k=1}^{\infty} \big(\phi_k, ze^{-(k-1)z}\phi_k\big)_{\mathcal{H}} = z\sum_{k=0}^{\infty} e^{-kz} = z\frac{1}{1-e^{-z}} = \frac{z}{e^z-1}e^z \underset{z\to 0}{\longrightarrow} 1.$$

$(ii)$ We shall now elaborate on the fact that an analytic function taking values in the space of bounded linear operators in a Hilbert space can indeed have different domains of analyticity if considered as taking values in particular Schatten–von Neumann ideals.

Consider an infinite-dimensional Hilbert space $\mathcal{H}$ and pick an orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$. For $z \in \mathbb{C}$ with $\operatorname{Re}(z) \geqslant 0$, define

$$T(z)\phi_k := e^{-z\ln(k)}\phi_k, \quad k \in \mathbb{N}.$$

Then $T(z) \in \mathcal{B}(\mathcal{H})$ and

$$T(z)\psi := \sum_{k=1}^{\infty} e^{-z\ln(k)}\, (\phi_k, \psi)\, \phi_k, \quad \psi \in \mathcal{H}.$$

Moreover, $T(0) = I_{\mathcal{H}}$, $T(2) \in \mathcal{B}_1(\mathcal{H})$, $T(1) \in \mathcal{B}_2(\mathcal{H})$. We note here that for $\operatorname{Re}(z) > 1$ the function $z \mapsto T(z)$ is also analytic with values in $\mathcal{B}_1(\mathcal{H})$, however, the trace norm of $T(\cdot)$ blows up at $z = 1$. $\diamond$

We conclude this section with some facts on Fredholm operators. For reasons to be able to handle certain classes of unbounded Fredholm operators in a convenient manner, we now take a slightly more general approach and permit a two-Hilbert space setting as follows: Suppose $\mathcal{H}_j$, $j \in \{1,2\}$, are complex, separable Hilbert spaces. Then $S\colon \operatorname{dom}(S) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, $S$ is called a *Fredholm operator*, denoted by $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, if

$(i)$ $S$ is closed and densely defined in $\mathcal{H}_1$.

$(ii)$ $\operatorname{ran}(S)$ is closed in $\mathcal{H}_2$.

$(iii)$ $\dim(\ker(S)) + \dim(\ker(S^*)) < \infty$.

If $S$ is Fredholm, its *Fredholm index* is given by

$$\operatorname{ind}(S) = \dim(\ker(S)) - \dim(\ker(S^*)). \tag{3.7}$$

If $S\colon \operatorname{dom}(S) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ is densely defined and closed, we associate with $\operatorname{dom}(S) \subset \mathcal{H}_1$ the standard graph Hilbert subspace $\mathcal{H}_S \subseteq \mathcal{H}_1$ induced by $S$ defined by

$$\begin{aligned}\mathcal{H}_S = (\operatorname{dom}(S); (\,\cdot\,,\,\cdot\,)_{\mathcal{H}_S}), \quad & (f,g)_{\mathcal{H}_S} = (Sf, Sg)_{\mathcal{H}_2} + (f,g)_{\mathcal{H}_1}, \\ & \|f\|_{\mathcal{H}_S} = \big[\|Sf\|_{\mathcal{H}_2}^2 + \|f\|_{\mathcal{H}_1}^2\big]^{1/2},\ f,g \in \operatorname{dom}(S).\end{aligned}$$

In addition, for $A_0, A_1 \in \Phi(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $A_0$ and $A_1$ are called *homotopic* in $\Phi(\mathcal{H}_1, \mathcal{H}_2)$ if there exists $A\colon [0,1] \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ continuous such that $A(t) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, $t \in [0,1]$, with $A(0) = A_0$, $A(1) = A_1$.

Next, following [11, Chs. 1, 3], [54, Chs. XI, XVII], [56, Sects. IV.6, IV.10], [77, Sect. I.3], [78, Ch. 2], [90, Chs. 5, 7], we now summarize a few basic properties of Fredholm operators.

**Theorem 3.6.** *Let $\mathcal{H}_j$, $j = 1, 2, 3$, be complex, separable Hilbert spaces, then the following items $(i)$–$(vii)$ hold:*

$(i)$ *If $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and $T \in \Phi(\mathcal{H}_2, \mathcal{H}_3)$, then $TS \in \Phi(\mathcal{H}_1, \mathcal{H}_3)$ and*

$$\operatorname{ind}(TS) = \operatorname{ind}(T) + \operatorname{ind}(S). \tag{3.8}$$

$(ii)$ *Assume that $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and $K \in \mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$, then $(S + K) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and*

$$\operatorname{ind}(S + K) = \operatorname{ind}(S). \tag{3.9}$$

$(iii)$ *Suppose that $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and $K \in \mathcal{B}_\infty(\mathcal{H}_S, \mathcal{H}_2)$, then $(S+K) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and*

$$\operatorname{ind}(S + K) = \operatorname{ind}(S). \tag{3.10}$$

$(iv)$ *Assume that $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$. Then there exists $\varepsilon(S) > 0$ such that for any $R \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with $\|R\|_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)} < \varepsilon(S)$, one has $(S + R) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and*

$$\operatorname{ind}(S + R) = \operatorname{ind}(S), \quad \dim(\ker(S + R)) \leqslant \dim(\ker(S)). \tag{3.11}$$

$(v)$ *Let $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, then $S^* \in \Phi(\mathcal{H}_2, \mathcal{H}_1)$ and*

$$\operatorname{ind}(S^*) = -\operatorname{ind}(S). \tag{3.12}$$

$(vi)$ *Assume that $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and that the Hilbert space $\mathcal{V}_1$ is continuously embedded in $\mathcal{H}_1$, with $\operatorname{dom}(S)$ dense in $\mathcal{V}_1$. Then $S \in \Phi(\mathcal{V}_1, \mathcal{H}_2)$ with $\ker(S)$ and $\operatorname{ran}(S)$ the same whether $S$ is viewed as an operator $S\colon \operatorname{dom}(S) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, or as an operator $S\colon \operatorname{dom}(S) \subseteq \mathcal{V}_1 \to \mathcal{H}_2$.*

$(vii)$ *Assume that the Hilbert space $\mathcal{W}_1$ is continuously and densely embedded in $\mathcal{H}_1$. If $S \in \Phi(\mathcal{W}_1, \mathcal{H}_2)$ then $S \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ with $\ker(S)$ and $\operatorname{ran}(S)$ the same whether $S$ is viewed as an operator $S\colon \operatorname{dom}(S) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, or as an operator $S\colon \operatorname{dom}(S) \subseteq \mathcal{W}_1 \to \mathcal{H}_2$.*

$(viii)$ *Homotopic operators in $\Phi(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ have equal Fredholm index. More precisely, the set $\Phi(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is open in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, hence $\Phi(\mathcal{H}_1, \mathcal{H}_2)$ contains at most countably many connected components, on each of which the Fredholm index is constant. Equivalently, $\operatorname{ind}\colon \Phi(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$ is locally constant, hence continuous, and homotopy invariant.*

A prime candidate for the Hilbert spaces $\mathcal{V}_1, \mathcal{W}_1 \subseteq \mathcal{H}_1$ in Theorem 3.6 $(vi), (vii)$ (e.g., in applications to differential operators) is the graph Hilbert space $\mathcal{H}_S$ induced by $S$. Moreover, an immediate consequence of Theorem 3.6 we will apply later is the following homotopy invariance of the Fredholm index for a family of Fredholm operators with fixed domain.

**Corollary 3.7.** *Let $T(s) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, $s \in I$, where $I \subseteq \mathbb{R}$ is a connected interval, with $\operatorname{dom}(T(s)) := \mathcal{V}_T$ independent of $s \in I$. In addition, assume that $\mathcal{V}_T$ embeds densely and continuously into $\mathcal{H}_1$ (for instance, $\mathcal{V}_T = \mathcal{H}_{T(s_0)}$ for some fixed $s_0 \in I$) and that $T(\cdot)$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{B}(\mathcal{V}_T, \mathcal{H}_2)}$. Then*

$$\operatorname{ind}(T(s)) \in \mathbb{Z} \text{ is independent of } s \in I. \tag{3.13}$$

The corresponding case of unbounded operators with varying domains (and $\mathcal{H}_1 = \mathcal{H}_2$) is treated in detail in [37].

## 4. On Schatten–von Neumann Classes and Trace Class Estimates

This is the first of two technical sections, providing basic results used later on in our detailed study of Dirac-type operators to be introduced in Section 6. We also recall results on the Schatten–von Neumann classes and apply these to concrete situations needed in Section 7.

We start with the following well-known characterization of Hilbert–Schmidt operators $\mathcal{B}_2\big(L^2(\Omega; d\mu)\big)$ in $L^2(\Omega; d\mu)$:

**Theorem 4.1** (see, e.g., [92, Theorem 2.11])**.** *Let $(\Omega; \mathcal{B}; \mu)$ be a separable measure space and $k\colon \Omega \times \Omega \to \mathbb{C}$ be $\mu \otimes \mu$ measurable. Then the map*

$$\mathcal{U} : \begin{cases} L^2(\Omega \times \Omega; d\mu \otimes d\mu) \to \mathcal{B}_2\big(L^2(\Omega; d\mu)\big), \\ k \mapsto \big(f \mapsto \big(\Omega \ni x \mapsto \int_\Omega k(x,y) f(y)\, d\mu(y)\big)\big), \end{cases}$$

*is unitary.*

The Hölder inequality is also valid for trace ideals with $p$-summable singular values.

**Theorem 4.2** (Hölder inequality, see, e.g., [92, Theorem 2.8])**.** *Assume that $\mathcal{H}$ is a complex, separable Hilbert space, $m \in \mathbb{N}$, $q_j \in [1, \infty]$, $j \in \{1, \dots, m\}$, $p \in [1, \infty]$. Assume that*

$$\sum_{j=1}^m \frac{1}{q_j} = \frac{1}{p}.$$

*Let $T_j \in \mathcal{B}_{q_j}(\mathcal{H})$, $j \in \{1, \dots, m\}$. Then $T \coloneqq \prod_{j=1}^m T_j \in \mathcal{B}_p(\mathcal{H})$ and*

$$\|T\|_{\mathcal{B}_p(\mathcal{H})} \leqslant \prod_{j=1}^m \|T_j\|_{\mathcal{B}_{q_j}(\mathcal{H})}.$$

For $q_1 = q_2 = m = 2$, one obtains a criterion for operators belonging to the trace class $\mathcal{B}_1$, which we shall use later on.

**Corollary 4.3.** *Let $(\Omega; \mathcal{B}, \mu)$ be a separable measure space, $k\colon \Omega \times \Omega \to \mathbb{C}$ be $\mu \otimes \mu$ measurable. Moreover, assume that there exists $\ell, m \in L^2(\Omega \times \Omega; d\mu \otimes d\mu)$ such that*

$$k(x,y) = \int_\Omega \ell(x,w) m(w,y)\, d\mu(w) \ \text{ for } \mu \otimes \mu \text{ a.e. } (x,y) \in \Omega \times \Omega.$$

*Then $K$, the associated integral operator with integral kernel $k(\cdot,\cdot)$ in $L^2(\Omega \times \Omega; d\mu \otimes d\mu)$, is trace class, $K \in \mathcal{B}_1\big(L^2(\Omega; d\mu)\big)$, and*

$$\operatorname{tr}_{L^2(\Omega; d\mu)}(K) = \int_\Omega k(x,x)\, d\mu(x) = \int_\Omega \int_\Omega \ell(x,w) m(w,x)\, d\mu(w) d\mu(x).$$

*Proof.* By Theorem 4.1 the integral operators $L$ and $M$ associated with $\ell$ and $m$, respectively, are Hilbert–Schmidt operators. Since $K = LM$, one gets $K \in \mathcal{B}_1\big(L^2(\Omega; d\mu)\big)$ by Theorem 4.2. Moreover, by Theorem 4.1, one concludes that

$$\begin{aligned} \operatorname{tr}_{L^2(\Omega; d\mu)}(K) &= \operatorname{tr}_{L^2(\Omega; d\mu)}(LM) = \operatorname{tr}_{L^2(\Omega; d\mu)}\big((L^*)^* M\big) \\ &= \int_\Omega \int_\Omega \overline{\overline{\ell(w,x)}} m(x,w)\, d\mu(x) d\mu(w) \\ &= \int_\Omega \int_\Omega \ell(x,w) m(w,x)\, d\mu(w) d\mu(x). \end{aligned}$$

$\square$

In the bulk of this manuscript, Theorem 4.1 and Corollary 4.3 will be applied to the case

$$L^2(\Omega;\mathcal{B};d\mu)=L^2(\mathbb{R}^n;\mathcal{B}(\mathbb{R}^n);d^nx)=L^2(\mathbb{R}^n)$$

We recall $H^1(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n)$, the spaces of once and twice weakly differentiable $L^2$-functions with derivatives in $L^2$, respectively. Moreover, we shall furthermore consider the differential operator $Q$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ by

$$Q\coloneqq\sum_{j=1}^n\gamma_{j,n}\partial_j,\quad \operatorname{dom}(Q)=H^1(\mathbb{R}^n)^{2^{\widehat{n}}},\tag{4.1}$$

where

$$\gamma_{j,n}^*=\gamma_{j,n}\in\mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}\ \text{ if } n=2\widehat{n} \text{ or } n=2\widehat{n}+1,\tag{4.2}$$

and $\gamma_{j,n}\gamma_{k,n}+\gamma_{k,n}\gamma_{j,n}=2\delta_{jk}$ for all $j,k\in\{1,\dots,n\}$, see Definition A.3. A first consequence is,

$$Q^2=\Delta I_{2^{\widehat{n}}},\quad \operatorname{dom}(Q^2)=H^2(\mathbb{R}^n)^{2^{\widehat{n}}}.\tag{4.3}$$

Indeed,

$$\begin{aligned}QQ&=\sum_{j=1}^n\gamma_{j,n}\partial_j\sum_{k=1}^n\gamma_{k,n}\partial_k=\sum_{j=1}^n\sum_{k=1}^n\gamma_{j,n}\partial_j\gamma_{k,n}\partial_k\\&=\frac12\sum_{j=1}^n\sum_{k=1}^n\gamma_{j,n}\partial_j\gamma_{k,n}\partial_k+\frac12\sum_{j=1}^n\sum_{k=1}^n\gamma_{k,n}\partial_k\gamma_{j,n}\partial_j\\&=\frac12\sum_{j=1}^n\sum_{k=1}^n\left(\gamma_{j,n}\gamma_{k,n}+\gamma_{k,n}\gamma_{j,n}\right)\partial_j\partial_k=\Delta I_{2^{\widehat{n}}}.\end{aligned}$$

We study the operator asssociated with the differential expression $Q$ with its properties later on in Section 6. More precisely, in Theorem 6.4 we show that $Q=-Q^*$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ with domain $H^1(\mathbb{R}^n)^{2^{\widehat{n}}}$, that is, $Q$ is skew-self-adjoint in $L^2\left(\mathbb{R}^n\right)^{2^{\widehat{n}}}$. In particular, we get for any $\mu\in\mathbb{C}$ with $\operatorname{Re}(\mu)\neq 0$ that

$$\left\|(Q+\mu)^{-1}\right\|\leqslant|\operatorname{Re}(\mu)|^{-1}\tag{4.4}$$

as an operator from $L^2\left(\mathbb{R}^n\right)^{2^{\widehat{n}}}$ to $L^2\left(\mathbb{R}^n\right)^{2^{\widehat{n}}}$. One notes that by Fourier transform, the operator $Q$ is unitarily equivalent to the Fourier multiplier with symbol $\sum_{j=1}^n\gamma_{j,n}(-i)\xi_j$. Furthermore, by $\gamma_{j,n}=\gamma_{j,n}^*$ and $\gamma_{j,n}^2=I_{2^{\widehat{n}}}$, the matrix $\gamma_{j,n}$ is unitary. Hence, the symbol of $Q$ may be estimated as follows

$$\left\|\sum_{j=1}^n\gamma_{j,n}(-i)\xi_j\right\|_{\mathcal{B}(\mathbb{C}^{2^{\widehat{n}}})}\leqslant\sum_{j=1}^n|\xi_j|\leqslant\sqrt{n}\left(\sum_{j=1}^n|\xi_j|^2\right)^{1/2}=\sqrt{n}\,|\xi|\,,\quad \xi\in\mathbb{R}^n.\tag{4.5}$$

We denote

$$R_\mu\coloneqq(-\Delta+\mu)^{-1},\quad \mu\in\mathbb{C}\backslash(-\infty,0].\tag{4.6}$$

We recall our notational conventions collected in Section 2. In particular, we recall $[A,B]=AB-BA$, the commutator of two operators $A$ and $B$, see also (6.15).

**Lemma 4.4.** *Let $\mu\in\mathbb{C}$, $\operatorname{Re}(\mu)>0$, and $\Psi\in C_b^2(\mathbb{R}^n)$. Then with $Q$ and $R_\mu$ given by* (4.1) *and* (4.6), *respectively, one obtains* (*cf. Remark* 2.1),

$$[R_\mu,\Psi]=R_\mu\left(Q^2\Psi\right)R_\mu+2R_\mu\left(Q\Psi\right)QR_\mu.\tag{4.7}$$

*Proof.* Recalling Remark 2.1 concerning multiplication operators, we compute with the help of (4.3)

$$\begin{aligned}
[R_\mu, \Psi] &= R_\mu \Psi - \Psi R_\mu \\
&= R_\mu \left(\Psi \left(-\Delta + \mu\right) - \left(-\Delta + \mu\right) \Psi\right) R_\mu \\
&= R_\mu \left(\Delta\Psi - \Psi\Delta\right) R_\mu = R_\mu \left(Q^2\Psi - \Psi\Delta\right) R_\mu \\
&= R_\mu \left(Q\left(Q\Psi\right) + Q\Psi Q - \Psi\Delta\right) R_\mu \\
&= R_\mu \left(\left(Q^2\Psi\right) + \left(Q\Psi\right) Q + \left(Q\Psi\right) Q + \Psi Q^2 - \Psi\Delta\right) R_\mu \\
&= R_\mu \left(\left(Q^2\Psi\right) + 2\left(Q\Psi\right) Q\right) R_\mu. \qquad \square
\end{aligned}$$

In the course of computing the index of the closed operator $L$ to be introduced later on, we need to establish trace class properties of operators that are products of operators of the form discussed in the following lemma. For given $n \in \mathbb{N}_{\geqslant 3}$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{C}_{\mathrm{Re}>0} \coloneqq \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$, we let

$$g_\mu(x) \coloneqq \frac{1}{\mathrm{Re}(\mu) + |x|^2}, \quad \widetilde{g}_\mu(x) \coloneqq \sqrt{n}\frac{|x|}{\mathrm{Re}(\mu) + |x|^2}. \tag{4.8}$$

One notes that $g_\mu \in L^q(\mathbb{R}^n)$ for all $q > n/2$ and $\widetilde{g}_\mu \in L^q(\mathbb{R}^n)$ for all $q > n$.

**Lemma 4.5.** *Let $\mu \in \mathbb{C}_{\mathrm{Re}>0}$, $\Psi \in L^\infty(\mathbb{R}^n)$, $\alpha \in [1, \infty)$, $n \geqslant 3$, and recall $R_\mu$ from (4.6) and $Q$ from (4.1), as well as $g_\mu$ and $\widetilde{g}_\mu$ from (4.8). Assume that there exists $\kappa > 0$ such that*

$$|\Psi(x)| \leqslant \kappa(1 + |x|)^{-\alpha} \ \text{ for a.e. } x \in \mathbb{R}^n.$$

*(i) Then for all $q > n$, $R_\mu\Psi, \Psi R_\mu \in \mathcal{B}_q\left(L^2\left(\mathbb{R}^n\right)\right)$ and*

$$\max\left( \|\Psi R_\mu\|_{\mathcal{B}_q(L^2(\mathbb{R}^n))}, \|R_\mu \Psi\|_{\mathcal{B}_q(L^2(\mathbb{R}^n))} \right) \leqslant (2\pi)^{-n/q} \|\Psi\|_{L^q(\mathbb{R}^n)} \|g_\mu\|_{L^q(\mathbb{R}^n)} < \infty.$$

*The assertion remains the same, if $R_\mu$ is replaced by $R_\mu Q$ or $QR_\mu$ and $\|g_\mu\|_{L^q(\mathbb{R}^n)}$ in the latter estimate is replaced by $\|\widetilde{g}_\mu\|_{L^q(\mathbb{R}^n)}$.*

*(ii) Assume, in addition, $\alpha > 3/2$. Then, if $n > 3$, there exists $\vartheta \in (3/4, 1)$ such that $R_\mu\Psi, \Psi R_\mu \in \mathcal{B}_{2n\vartheta/3}\big(L^2(\mathbb{R}^n)\big)$. Moreover,*

$$\begin{aligned}
&\max\left( \|\Psi R_\mu\|_{\mathcal{B}_{2n\vartheta/3}(L^2(\mathbb{R}^n))}, \|R_\mu \Psi\|_{\mathcal{B}_{2n\vartheta/3}(L^2(\mathbb{R}^n))} \right) \\
&\quad \leqslant (2\pi)^{-3/(2\vartheta)} \|\Psi\|_{L^{2n\vartheta/3}(\mathbb{R}^n)} \|g_\mu\|_{L^{2n\vartheta/3}(\mathbb{R}^n)} < \infty.
\end{aligned}$$

*For $n = 3$, $R_\mu\Psi, \Psi R_\mu \in \mathcal{B}_2\left(L^2\left(\mathbb{R}^n\right)\right)$ and*

$$\max\left( \|\Psi R_\mu\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))}, \|R_\mu \Psi\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))} \right) \leqslant (2\pi)^{-3/2} \|\Psi\|_{L^2(\mathbb{R}^n)} \|g_\mu\|_{L^2(\mathbb{R}^n)} < \infty.$$

*(iii) Let $\Theta \in C_b^2\left(\mathbb{R}^n\right)$ with*

$$\left|\left(Q\Theta\right)(x)\right| + \left|\left(Q^2\Theta\right)(x)\right| \leqslant \kappa(1 + |x|)^{-\beta}$$

*for some $\kappa > 0$ and $\beta > 3/2$. Then, recalling (2.2), $[R_\mu, \Theta] \in \mathcal{B}_2\left(L^2\left(\mathbb{R}^n\right)\right)$ with*

$$\begin{aligned}
&\|[R_\mu, \Theta]\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))} \\
&\quad \leqslant \left( \frac{1}{\mathrm{Re}(\mu)} + 2\left( \frac{1}{\mathrm{Re}(\sqrt{\mu})} + 3\frac{\left|\sqrt{\mu}\right|}{\mathrm{Re}(\mu)} \right) \right) (2\pi)^{-3/2} \\
&\qquad \times \max\left( \|Q^2\Theta\|_{L^2(\mathbb{R}^n)}, \|Q\Theta\|_{L^2(\mathbb{R}^n)} \right) \|g_\mu\|_{L^2(\mathbb{R}^n)} < \infty
\end{aligned}$$

*if* $n = 3$, *and* $[R_\mu, \Theta] \in \mathcal{B}_{(2n/3)\vartheta}\left(L^2\left(\mathbb{R}^n\right)\right)$ *with*

$$\begin{aligned}\|[R_\mu, \Theta]\|_{\mathcal{B}_{2n\vartheta/3}(L^2(\mathbb{R}^n))} \leqslant & \left(\frac{1}{\operatorname{Re}(\mu)} + 2\left(\frac{1}{\operatorname{Re}(\sqrt{\mu})} + 3\frac{|\sqrt{\mu}|}{\operatorname{Re}(\mu)}\right)\right)(2\pi)^{-3/(2\vartheta)} \\ & \times \max\left(\|Q^2\Theta\|_{L^{2n\vartheta/3}(\mathbb{R}^n)}, \|Q\Theta\|_{L^{2n\vartheta/3}(\mathbb{R}^n)}\right) \\ & \times \|g_\mu\|_{L^{2n\vartheta/3}(\mathbb{R}^n)} < \infty\end{aligned}$$

*for some* $\vartheta \in (0,1)$ *with* $2n\vartheta/3 \geqslant 2$, *if* $n > 3$.

The proof of Lemma 4.5, is basically contained in the following result:

**Theorem 4.6** ([92, Theorem 4.1])**.** *Let* $n \in \mathbb{N}$, $p \geqslant 2$, *and* $\Psi, g \in L^p(\mathbb{R}^n)$. *Define* $T_{\Psi,g} \in \mathcal{B}(L^2(\mathbb{R}^n))$ *as the operator of composition of multiplication by* $\Psi$ *and* $g(i\partial_1, \dots, i\partial_n)$ *as a Fourier multiplier. Then* $T_{\Psi,g} \in \mathcal{B}_p(L^2(\mathbb{R}^n))$ *and*

$$\|T_{\Psi,g}\|_{\mathcal{B}_p(L^2(\mathbb{R}^n))} \leqslant (2\pi)^{-n/p}\|\Psi\|_{L^p(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}. \tag{4.9}$$

The proof of (4.9) rests on the observation that there is equality for $p = 2$ and a straight forward estimate for the limiting case $p = \infty$. The general case follows via complex interpolation.

*Proof of Lemma* 4.5. Observing $\|T\|_{\mathcal{B}_p(\mathcal{H})} = \|T^*\|_{\mathcal{B}_p(\mathcal{H})}$ for all $T \in \mathcal{B}_p(\mathcal{H})$, we shall only show the respective assertions for $\Psi R_\mu$. For parts $(i)$ and $(ii)$ one uses Theorem 4.6: One notes that for $\phi_\mu(\xi) \coloneqq \left(|\xi|^2 + \mu\right)^{-1}$, $\phi_\mu\left(i\partial_1, \dots, i\partial_n\right) = R_\mu$. Moreover, one observes that $|\phi_\mu| \leqslant g_\mu \in L^p\left(\mathbb{R}^n\right)$ for all $p > n/2$. In order to prove item $(i)$ one notes that $\alpha \geqslant 1$ implies that $\Psi \in L^q\left(\mathbb{R}^n\right)$ for all $q > n$. Hence, $\Psi R_\mu \in \mathcal{B}_q\big(L^2(\mathbb{R}^n)\big)$. The remaining assertion follows from the fact that the Fourier transform of $QR_\mu$ lies in $L^q$ as it can be estimated by $\widetilde{g}_\mu$, see (4.5). For part $(ii)$ one first considers the case $n = 3$. Then $2\alpha = \alpha(2/3)3 > 3 = n$. Hence, $\Psi \in L^2\left(\mathbb{R}^3\right)$ and, since $2 > 3/2 = n/2$, one infers that $\Psi R_\mu \in \mathcal{B}_2$. If $n > 3$, there exists $\vartheta \in (3/4, 1)$ such that $\alpha\vartheta > 3/2$. In particular, one has $(2/3)n\vartheta > (2/3)4(3/4) = 2$. Since $\alpha(2/3)\vartheta n > (3/2)\,(2n/3) = n$, one gets $\Psi \in L^{2n\vartheta/3}\left(\mathbb{R}^n\right)$. Moreover, since $(2n/3)\,\vartheta > n/2$ as $\vartheta > 3/4$, one concludes that $|\phi_\mu| \leqslant g_\mu \in L^{2n\vartheta/3}\left(\mathbb{R}^n\right)$, implying $\Psi R_\mu \in \mathcal{B}_{2n\vartheta/3}$. In order to show part $(iii)$ one notes that Lemma 4.4 implies

$$[R_\mu, \Theta] = R_\mu\left(Q^2\Theta\right)R_\mu + 2R_\mu\left(Q\Theta\right)QR_\mu.$$

Since $QR_\mu$ is a bounded linear operator, using (4.3) as well as (4.6), one deduces from

$$\begin{aligned}QR_\mu &= (Q + \sqrt{\mu})^{-1}\,(Q + \sqrt{\mu})\,(Q - \sqrt{\mu})\,R_\mu + \sqrt{\mu}R_\mu \\ &= (Q + \sqrt{\mu})^{-1} + \sqrt{\mu}R_\mu\end{aligned}$$

its corresponding norm bound $[\operatorname{Re}(\sqrt{\mu})]^{-1} + |\sqrt{\mu}|[\operatorname{Re}(\mu)]^{-1}$, see (4.4) for the norm bound of $\left(Q + \sqrt{\mu}\right)^{-1}$. Thus, the assertion follows from part $(ii)$ and the ideal property of the Schatten–von Neumann classes. □

Lemma 4.5 is decisive for obtaining the following result. We mention here that H. Vogt [99] subsequently managed to prove the following theorem in a direct way without using Lemma 4.5 and thus without the use of Theorem 4.6. In the following theorem (and throughout this manuscript later on) we recall our simplifying convention (2.1) to abbreviate finite operator products $A_1A_2\cdots A_N$ by $\prod_{j=1}^N A_j$,

regardless of underlying noncommutativity issues, upon relying on ideal properties of the bounded operators $A_j$, $j=1,\dots,N$, $N\in\mathbb{N}$.

**Theorem 4.7.** *Let $n=2\widehat{n}+1\in\mathbb{N}_{\geqslant 3}$ odd, $\Psi_1,\dots,\Psi_{\widehat{n}+1}\in C_b^2(\mathbb{R}^n)$, $\alpha_1,\dots,\alpha_{\widehat{n}+1}\in[1,\infty)$, $\varepsilon>1/2$, $\kappa>0$, $\mu\in\mathbb{C}_{\mathrm{Re}>0}$, and let $R_\mu$, $Q$, and $[\Psi_{\widehat{n}},R_\mu]$ be given by (4.6), (4.1), and (2.2), respectively.*

*$(i)$ Assume that for all $x\in\mathbb{R}^n$ and $j\in\{1,\dots,\widehat{n}+1\}$,*

$$|\Psi_j(x)|\leqslant\kappa(1+|x|)^{-\alpha_j}.$$

*Then*

$$\prod_{j=1}^{\widehat{n}+1}\Psi_jR_\mu,\prod_{j=1}^{\widehat{n}+1}R_\mu\Psi_j\in\mathcal{B}_2\big(L^2(\mathbb{R}^n)\big),$$

*and*

$$\bigg\|\prod_{j=1}^{\widehat{n}+1}\Psi_jR_\mu\bigg\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))}\leqslant\prod_{j=1}^{\widehat{n}+1}\|\Psi_jR_\mu\|_{\mathcal{B}_{n+1}(L^2(\mathbb{R}^n))}.$$

*$(ii)$ Assume for all $x\in\mathbb{R}^n$ and $j\in\{1,\dots,\widehat{n}-1\}$,*

$$|\Psi_j(x)|\leqslant\kappa(1+|x|)^{-\alpha_j},$$

*and*

$$|(Q\Psi_{\widehat{n}})(x)|+\big|\big(Q^2\Psi_{\widehat{n}}\big)(x)\big|\leqslant\kappa(1+|x|)^{-\alpha_{\widehat{n}}-\varepsilon}.$$

*Then*

$$\prod_{j=1}^{\widehat{n}-1}\Psi_jR_\mu\,[\Psi_{\widehat{n}},R_\mu]\in\mathcal{B}_2\big(L^2(\mathbb{R}^n)\big).$$

*Proof.* In order to prove parts $(i)$ and $(ii)$, we use Theorem 4.2 and Lemma 4.5. For part $(i)$ one observes that $\Psi_jR_\mu\in\mathcal{B}_{n+1}$ by Lemma 4.5 $(i)$ for all $j\in\{1,\dots,\widehat{n}+1\}$. Moreover, by $\sum_{j=1}^{\widehat{n}+1}\frac{1}{n+1}=\big(\frac{n-1}{2}+1\big)\frac{1}{n+1}=1/2$ one concludes with the help of Theorem 4.2 that $\prod_{j=1}^{\widehat{n}+1}\Psi_jR_\mu\in\mathcal{B}_2$.

In order to arrive at item $(ii)$, one notes that the case $n=3$ directly follows from Lemma 4.5 $(iii)$ since in that case $\widehat{n}-1=0$. For $n>3$ there exists $\vartheta\in(3/4,1)$ such that $[\Psi_{\widehat{n}},R_\mu]\in\mathcal{B}_{2n\vartheta/3}$. The assertion is clear if $2n\vartheta/3\leqslant 2$. Thus, we assume that $2n\vartheta/3>2$. Let $q\in\mathbb{R}\backslash\{0\}$ be such that

$$(\widehat{n}-1)\frac{1}{q}+\frac{1}{(2n/3)\vartheta}=\frac{1}{2}. \tag{4.10}$$

Equation (4.10) with $\widehat{n}-1=(n-3)/2$ reveals

$$\frac{1}{q}=\left(\frac{n\vartheta-3}{n\vartheta}\right)\frac{1}{n-3}=\left(\frac{n\vartheta-3\vartheta}{n\vartheta}\right)\frac{1}{n-3}+3\left(\frac{\vartheta-1}{\vartheta}\right)\frac{1}{n(n-3)}<\frac{1}{n}.$$

Hence, $q>n$ and, so, from $\Psi_jR_\mu\in\mathcal{B}_q$, by Lemma 4.5, the assertion follows from Theorem 4.2. □

In order to illustrate the latter mechanism and for later purposes, we now discuss an example.

**Example 4.8.** *Let $z > -1$, and $\Phi \in C_b^\infty(\mathbb{R}^3;\mathbb{C}^{2\times 2})$ such that for $x \in \mathbb{R}^3$, with $|x| \geqslant 1$,*

$$\Phi(x) = \sum_{j=1}^{3} \frac{x_j}{|x|}\sigma_j.$$

*Here $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ denote the Pauli matrices, see also Definition* A.3. *Recalling our convention* (2.5) *and $R_{1+z} = (-\Delta+1+z)^{-1}$, we now prove that the operator given by*

$$\begin{aligned} T := \operatorname{tr}_4 \Bigg( & \sum_{k=1}^{3} \big((R_{1+z}\sigma_k) \otimes (\partial_k \Phi)\big) \\ & \times \sum_{k=1}^{3} \big((R_{1+z}\sigma_k) \otimes (\partial_k \Phi)\big) \sum_{k=1}^{3} \big((R_{1+z}\sigma_k) \otimes (\partial_k \Phi)\big) R_{1+z} \Bigg) \end{aligned}$$

*is trace class, $T \in \mathcal{B}_1\big(L^2(\mathbb{R}^3)\big)$.*

*First of all, with the help of Proposition* A.8 *and introducing the fully anti-symmetric symbol in* 3 *coordinates, $\varepsilon_{jk\ell}$, $j,k,\ell \in \{1,2,3\}$, we may express $T$ as follows* (*for notational simplicity, we now drop all tensor product symbols*),

$$\begin{aligned} T &= \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} \operatorname{tr}_4 \big(\sigma_{k_1}\sigma_{k_2}\sigma_{k_3} R_{1+z} I_2 (\partial_{k_1}\Phi) R_{1+z} I_2 (\partial_{k_2}\Phi) R_{1+z} I_2 (\partial_{k_3}\Phi) R_{1+z} I_2\big) \\ &= \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} \operatorname{tr}_2 (\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}) \\ &\qquad \times \operatorname{tr}_2 (R_{1+z} I_2 (\partial_{k_1}\Phi) R_{1+z} I_2 (\partial_{k_2}\Phi) R_{1+z} I_2 (\partial_{k_3}\Phi) R_{1+z} I_2) \\ &= \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} I_2 (\partial_{k_1}\Phi) R_{1+z} I_2 (\partial_{k_2}\Phi) R_{1+z} I_2 (\partial_{k_3}\Phi) R_{1+z} I_2) \\ &= \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} (\partial_{k_1}\Phi) R_{1+z} (\partial_{k_2}\Phi) R_{1+z} (\partial_{k_3}\Phi) R_{1+z}) \\ &= \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} [(\partial_{k_1}\Phi), R_{1+z}] (\partial_{k_2}\Phi) R_{1+z} (\partial_{k_3}\Phi) R_{1+z}) \\ &\quad + \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} R_{1+z} (\partial_{k_1}\Phi)(\partial_{k_2}\Phi) R_{1+z} (\partial_{k_3}\Phi) R_{1+z}) \\ &= \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} [(\partial_{k_1}\Phi), R_{1+z}] (\partial_{k_2}\Phi) R_{1+z} (\partial_{k_3}\Phi) R_{1+z}) \\ &\quad + \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} R_{1+z} (\partial_{k_1}\Phi)(\partial_{k_2}\Phi) [R_{1+z}, (\partial_{k_3}\Phi)] R_{1+z}) \\ &\quad + \sum_{1\leqslant k_1,k_2,k_3 \leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 (R_{1+z} R_{1+z} (\partial_{k_1}\Phi)(\partial_{k_2}\Phi)(\partial_{k_3}\Phi) R_{1+z} R_{1+z}). \end{aligned}$$

*One computes for $k \in \{1,2,3\}$ and $x \in \mathbb{R}^3$, $|x| \geqslant 1$,*

$$(\partial_k \Phi)(x) = \sum_{j=1}^{3} \left( \frac{\delta_{kj}}{|x|} - \frac{x_j}{|x|^2}\frac{x_k}{|x|} \right) \sigma_j,$$

*and observes that* $\|(\partial_k\Phi)(x)\| \leqslant 6/|x|$. *Moreover, it is easy to see that for all* $\beta \in \mathbb{N}_0^3$, *with* $|\beta| \coloneqq \sum_{j=1}^3 \beta_j \geqslant 2$, *there exists* $\kappa > 0$ *such that for all* $x \in \mathbb{R}^n$,

$$\|(\partial^\beta\Phi)(x)\| \leqslant \kappa(1+|x|)^{-2}.$$

*The latter estimate together with Theorem 4.7 yields that* $T \in \mathcal{B}_1\big(L^2(\mathbb{R}^3)\big)$ *if and only if*

$$\begin{aligned}\widetilde{T} \coloneqq \sum_{1\leqslant k_1,k_2,k_3\leqslant 3} 2i\varepsilon_{k_1k_2k_3} \operatorname{tr}_2 \left(R_{1+z}R_{1+z}(\partial_{k_1}\Phi)(\partial_{k_2}\Phi)(\partial_{k_3}\Phi)R_{1+z}R_{1+z}\right)& \\ \in \mathcal{B}_1\big(L^2(\mathbb{R}^3)\big).&\end{aligned}$$

*The latter operator can be rewritten as*

$$\widetilde{T} = \sum_{1\leqslant k_1,k_2,k_3\leqslant 3} 2i\varepsilon_{k_1k_2k_3} R_{1+z}^2 \operatorname{tr}_2 \left((\partial_{k_1}\Phi)(\partial_{k_2}\Phi)(\partial_{k_3}\Phi)\right) R_{1+z}^2.$$

*Next, we inspect the term in the middle in more detail:*

$$\begin{aligned}&\sum_{1\leqslant k_1,k_2,k_3\leqslant 3} \varepsilon_{k_1k_2k_3} \operatorname{tr}_2 \left((\partial_{k_1}\Phi)(\partial_{k_2}\Phi)(\partial_{k_3}\Phi)\right) \\ &\quad = \operatorname{tr}_2\left((\partial_1\Phi)(\partial_2\Phi)(\partial_3\Phi)\right) - \operatorname{tr}_2\left((\partial_1\Phi)(\partial_3\Phi)(\partial_2\Phi)\right) + \operatorname{tr}_2\left((\partial_2\Phi)(\partial_3\Phi)(\partial_1\Phi)\right) \\ &\qquad - \operatorname{tr}_2\left((\partial_2\Phi)(\partial_1\Phi)(\partial_3\Phi)\right) + \operatorname{tr}_2\left((\partial_3\Phi)(\partial_1\Phi)(\partial_2\Phi)\right) - \operatorname{tr}_2\left((\partial_3\Phi)(\partial_2\Phi)(\partial_1\Phi)\right) \\ &\quad = 3\operatorname{tr}_2\left((\partial_1\Phi)(\partial_2\Phi)(\partial_3\Phi)\right) - 3\operatorname{tr}_2\left((\partial_1\Phi)(\partial_3\Phi)(\partial_2\Phi)\right).\end{aligned}$$

*Employing*

$$\begin{aligned}&(\partial_1\Phi)(\partial_2\Phi)(\partial_3\Phi)(x) \\ &\quad = \sum_{1\leqslant j_1,j_2,j_3\leqslant 3} \left(\frac{\delta_{1j_1}}{|x|} - \frac{x_{j_1}}{|x|^2}\frac{x_1}{|x|}\right)\sigma_{j_1}\left(\frac{\delta_{2j_2}}{|x|} - \frac{x_{j_2}}{|x|^2}\frac{x_2}{|x|}\right)\sigma_{j_2}\left(\frac{\delta_{3j_3}}{|x|} - \frac{x_{j_3}}{|x|^2}\frac{x_3}{|x|}\right)\sigma_{j_3},\end{aligned}$$

*one gets*

$$\begin{aligned}&\frac{1}{2i}|x|^3 \operatorname{tr}_2\left((\partial_1\Phi)(\partial_2\Phi)(\partial_3\Phi)\right)(x) \\ &\quad = \sum_{1\leqslant j_1,j_2,j_3\leqslant 3} \varepsilon_{j_1j_2j_3}\left(\delta_{1j_1} - \frac{x_{j_1}x_1}{|x|^2}\right)\left(\delta_{2j_2} - \frac{x_{j_2}x_2}{|x|^2}\right)\left(\delta_{3j_3} - \frac{x_{j_3}x_3}{|x|^2}\right) \\ &\quad = \left(1 - \frac{x_1x_1}{|x|^2}\right)\left(1 - \frac{x_2x_2}{|x|^2}\right)\left(1 - \frac{x_3x_3}{|x|^2}\right) \\ &\qquad - \left(1 - \frac{x_1x_1}{|x|^2}\right)\left(-\frac{x_3x_2}{|x|^2}\right)\left(-\frac{x_2x_3}{|x|^2}\right) + \left(-\frac{x_2x_1}{|x|^2}\right)\left(-\frac{x_3x_2}{|x|^2}\right)\left(-\frac{x_1x_3}{|x|^2}\right) \\ &\qquad - \left(-\frac{x_2x_1}{|x|^2}\right)\left(-\frac{x_1x_2}{|x|^2}\right)\left(1 - \frac{x_3x_3}{|x|^2}\right) + \left(-\frac{x_3x_1}{|x|^2}\right)\left(-\frac{x_1x_2}{|x|^2}\right)\left(-\frac{x_2x_3}{|x|^2}\right) \\ &\qquad - \left(-\frac{x_3x_1}{|x|^2}\right)\left(1 - \frac{x_2x_2}{|x|^2}\right)\left(-\frac{x_1x_3}{|x|^2}\right) \\ &\quad = 1 - \frac{x_1^2}{|x|^2} - \frac{x_2^2}{|x|^2} - \frac{x_3^2}{|x|^2} \\ &\qquad + \frac{x_1^2x_2^2}{|x|^4} + \frac{x_1^2x_3^2}{|x|^4} + \frac{x_2^2x_3^2}{|x|^4} - \frac{x_3x_2x_2x_3}{|x|^4} - \frac{x_2x_1x_1x_2}{|x|^4} - \frac{x_3x_1x_1x_3}{|x|^4} \\ &\qquad - \frac{x_1^2x_2^2x_3^2}{|x|^6} + \frac{x_1^2x_2^2x_3^2}{|x|^6} - \frac{x_1^2x_2^2x_3^2}{|x|^6} + \frac{x_1^2x_2^2x_3^2}{|x|^6} - \frac{x_1^2x_2^2x_3^2}{|x|^6} + \frac{x_1^2x_2^2x_3^2}{|x|^6}\end{aligned}$$

$$= 0.$$

*On the other hand,*

$$\begin{aligned}
&\frac{1}{2i}|x|^3 \operatorname{tr}_2\left((\partial_1\Phi)(\partial_3\Phi)(\partial_2\Phi)\right)(x) \\
&\quad = \sum_{1\leqslant j_1,j_2,j_3\leqslant 3} \varepsilon_{j_1j_2j_3}\left(\delta_{1j_1} - \frac{x_{j_1}x_1}{|x|^2}\right)\left(\delta_{3j_2} - \frac{x_{j_2}x_3}{|x|^2}\right)\left(\delta_{2j_3} - \frac{x_{j_3}x_2}{|x|^2}\right) = 0.
\end{aligned}$$

*We note that in this example, the corresponding formula* (1.22) *is in fact valid, for* $|x| \geqslant 1$. *This example is of a similar type as in* [22, Section IV]. *This may well be the reason for the erroneous statement in* [22, p. 226, 2nd highlighted formula].

We shall also use on occasion the following Hilbert–Schmidt criterion for exactly $\widehat{n} = (n-1)/2$ factors:

**Theorem 4.9.** *Let* $n = 2\widehat{n}+1 \in \mathbb{N}_{\geqslant 3}$ *odd, and assume that* $\Psi_1, \dots, \Psi_{\widehat{n}} \in L^\infty(\mathbb{R}^n)$, $\alpha_1, \dots, \alpha_{\widehat{n}} \in [1,\infty)$, $\mu \in \mathbb{C}$, $\operatorname{Re}(\mu) > 0$. *Let* $R_\mu$, $Q$ *be given by* (4.6) *and* (4.1), *respectively. Assume that* $\alpha_{j^*} > 3/2$ *for some* $j^* \in \{1, \dots, \widehat{n}\}$. *Then*

$$T := \Bigg(\prod_{j=1}^{j^*-1} \Psi_j R_\mu\Bigg)\Psi_{j^*}R_\mu\Bigg(\prod_{j=j^*+1}^{\widehat{n}} \Psi_j R_\mu\Bigg) \in \mathcal{B}_2\left(L^2(\mathbb{R}^n)\right),$$

*and*

$$\begin{aligned}
&\|T\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))} \\
&\quad \leqslant \begin{cases} \big(\prod_{j\in\{1,\dots,\widehat{n}\}\backslash\{j^*\}} \|\Psi_j R_\mu\|_{\mathcal{B}_q(L^2(\mathbb{R}^n))}\big)\|\Psi_j^* R_\mu\|_{\mathcal{B}_r(L^2(\mathbb{R}^n))}, & \widehat{n} > 1, \\ \|\Psi_1 R_\mu\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))}, & \widehat{n} = 1, \end{cases}
\end{aligned}$$

*where* $q = 2(\widehat{n}-1)\vartheta/(n\vartheta - 3) > n$ *and* $r = 2n\vartheta/3 > 2$ *for some* $\vartheta \in (3/4, 1)$, *according to Lemma* 4.5 *(ii)*.

*The assertion is the same if some of the factors with index* $j \in \{1, \dots, \widehat{n}\}\backslash\{j^*\}$ *in the expression for* $T$ *are replaced by* $\Psi_j Q R_\mu$.

*Proof.* By Lemma 4.5 *(ii)* one observes that for $\widehat{n} = j^* = 1$, $\Psi_1 R_\mu \in \mathcal{B}_2\big(L^2(\mathbb{R}^n)\big)$, and the assertion follows. The rest of the proof is similar to the one of the concluding lines of Theorem 4.7. $\square$

## 5. Pointwise Estimates for Integral Kernels

The proof of the index theorem relies on (pointwise) estimates of integral kernels of certain integral operators. These integral operators are of a form similar to the one in Theorem 4.7. In order to guarantee that point-evaluation is a well-defined operation, these operators have to possess certain smoothing properties. Before proving the corresponding result, we define the Dirac $\delta$-distribution of point-evaluation at some point $x \in \mathbb{R}^n$ of a suitable function $f\colon \mathbb{R}^n \to \mathbb{C}$ by

$$\delta_{\{x\}} f \coloneqq f(x).$$

We note that for every $x \in \mathbb{R}^n$ one has $\delta_{\{x\}} \in H^{-(n/2)-\varepsilon}(\mathbb{R}^n)$ for all $\varepsilon > 0$, by the Sobolev embedding theorem (see, e.g., [2, Theorem 7.34(c)]), and recall that

$$H^s(\mathbb{R}^n) \coloneqq \big\{ f \in L^2(\mathbb{R}^n) \,\big|\, (1 + |\cdot|^2)^{s/2} (\mathcal{F} f) \in L^2(\mathbb{R}^n) \big\}, \quad s \in \mathbb{R}, \tag{5.1}$$

with norm denoted by $\|\cdot\|_{H^s(\mathbb{R}^n)}$, where $\mathcal{F}$ denotes the (distributional) Fourier transform being an extension of

$$(\mathcal{F}\phi)(x) \coloneqq (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ixy} \phi(y)\, d^n y, \quad x \in \mathbb{R}^n,\ \phi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{5.2}$$

For $f \in H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n)$, we will find it convenient to write

$$\big\langle \delta_{\{x\}}, f \big\rangle_{L^2(\mathbb{R}^n)} \coloneqq \delta_{\{x\}} f = f(x), \tag{5.3}$$

where $\langle \cdot\,, \cdot \rangle_{L^2(\mathbb{R}^n)}$ is understood as the continuous extension of the scalar product on $L^2(\mathbb{R}^n)$ to the pairing on the entire fractional-order Sobolev scale, $\langle \cdot\,, \cdot \rangle_{L^2(\mathbb{R}^n)} \coloneqq \langle \cdot\,, \cdot \rangle_{H^{-s}(\mathbb{R}^n), H^s(\mathbb{R}^n)}$, $s \geqslant 0$.

For convenience of the reader we now prove the following known result:

**Theorem 5.1.** *Let $n \in \mathbb{N}$, $\varepsilon > 0$, $T\colon H^{-(n/2)-\varepsilon}(\mathbb{R}^n) \to H^{(n/2)+\varepsilon}(\mathbb{R}^n)$ linear and bounded (cf.* (5.1) *and* (5.3)*). Then the map*

$$t\colon \mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto t(x,y) = \big\langle \delta_{\{x\}}, T\delta_{\{y\}} \big\rangle_{L^2(\mathbb{R}^n)} \in \mathbb{C}$$

*is well-defined, continuous, and bounded. Moreover, if*

$$T \in \mathcal{B}\big(H^{-(n/2)-1-\varepsilon}(\mathbb{R}^n), H^{(n/2)+\varepsilon}(\mathbb{R}^n)\big) \cap \mathcal{B}\big(H^{-(n/2)-\varepsilon}(\mathbb{R}^n), H^{(n/2)+1+\varepsilon}(\mathbb{R}^n)\big), \tag{5.4}$$

*then $t$ is bounded and continuously differentiable with bounded derivatives $\partial_j t$, $j \in \{1, \dots, 2n\}$.*

*Remark* 5.2. We note that with the maps and assumptions introduced in Theorem 5.1, $t(\cdot,\cdot)$ is in fact the *integral kernel* of $T$, that is, $t$ satisfies

$$(Tf)(x) = \int_{\mathbb{R}^n} t(x,y) f(y)\, d^n y, \quad x \in \mathbb{R}^n,$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Indeed, let $x \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then one computes

$$\begin{aligned}
(Tf)(x) &= \big\langle \delta_{\{x\}}, Tf \big\rangle_{L^2(\mathbb{R}^n)} = \big\langle T^*\delta_{\{x\}}, f \big\rangle_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \overline{T^*\delta_{\{x\}}(y)} f(y)\, d^n y = \int_{\mathbb{R}^n} \overline{\big\langle \delta_{\{y\}}, T^*\delta_{\{x\}} \big\rangle_{L^2(\mathbb{R}^n)}} f(y)\, d^n y \\
&= \int_{\mathbb{R}^n} \overline{\big\langle T\delta_{\{y\}}, \delta_{\{x\}} \big\rangle_{L^2(\mathbb{R}^n)}} f(y)\, d^n y = \int_{\mathbb{R}^n} \big\langle \delta_{\{x\}}, T\delta_{\{y\}} \big\rangle_{L^2(\mathbb{R}^n)} f(y)\, d^n y \\
&= \int_{\mathbb{R}^n} t(x,y) f(y)\, d^n y.
\end{aligned}$$

◇

*Proof of Theorem* 5.1. First, one observes that for any $y \in \mathbb{R}^n$, $\delta_{\{y\}} \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$ and $\mathcal{F}\delta_{\{y\}}(x) = (2\pi)^{-n/2} e^{ixy}$ for all $x, y \in \mathbb{R}^n$. Consequently, $T\delta_{\{y\}} \in H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n)$, and hence $\langle \delta_{\{x\}}, T\delta_{\{y\}} \rangle_{L^2(\mathbb{R}^n)}$ is well-defined for all $x, y \in \mathbb{R}^n$. Moreover, from

$$\begin{aligned}\left|\langle \delta_{\{x\}}, T\delta_{\{y\}} \rangle_{L^2(\mathbb{R}^n)}\right| \leqslant \left|\delta_{\{x\}}\right|_{-\frac{n}{2}-\varepsilon} \left|T\delta_{\{y\}}\right|_{\frac{n}{2}+\varepsilon} &\leqslant \left|\delta_{\{0\}}\right|_{-\frac{n}{2}-\varepsilon} \left|\delta_{\{0\}}\right|_{-\frac{n}{2}-\varepsilon} \\ &\quad \times \|T\|_{\mathcal{B}\left(H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n), H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n)\right)}, \quad x, y \in \mathbb{R}^n,\end{aligned}$$

one concludes the boundedness of $t$. Next, we show sequential continuity of $t$. One observes that by the Sobolev embedding theorem, the map $T\delta_{\{y\}}$ is continuous for all $y \in \mathbb{R}^n$ (One recalls that $\mathcal{F}T\delta_{\{y\}} \in L^1(\mathbb{R}^n)$ with the Fourier transform given by (5.2).) Let $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}^n \times \mathbb{R}^n$ and denote its limit as $k \to \infty$ by $(x, y)$. One notes that $\left|\delta_{\{y\}} - \delta_{\{y_k\}}\right|_{-\frac{n}{2}-\varepsilon} \to 0$, as $k \to \infty$. Indeed, one gets by Lebesgue's dominated convergence theorem that

$$\left|\delta_{\{y\}} - \delta_{\{y_k\}}\right|^2_{-\frac{n}{2}-\varepsilon} = (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\left(e^{ixy} - e^{ixy_k}\right)\right|^2 \left[1 + |x|^2\right]^{-(n/2)-\varepsilon} d^n x \underset{k\to\infty}{\longrightarrow} 0.$$

Moreover, one observes that $\{\delta_{\{x_k\}}\}_{k\in\mathbb{N}}$ is uniformly bounded in $H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$ by some constant $M$. Next, let $\eta > 0$ and choose $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$ one has $\left|\delta_{\{y\}} - \delta_{\{y_k\}}\right|_{-\frac{n}{2}-\varepsilon} \leqslant \eta$ and $\left|T\delta_{\{y\}}(x_k) - T\delta_{\{y\}}(x)\right| \leqslant \eta$. Then one estimates for $k \in \mathbb{N}$,

$$\begin{aligned}&\left|\langle \delta_{\{x_k\}}, T\delta_{\{y_k\}} \rangle_{L^2(\mathbb{R}^n)} - \langle \delta_{\{x\}}, T\delta_{\{y\}} \rangle_{L^2(\mathbb{R}^n)}\right| \\ &\quad \leqslant \left|\langle \delta_{\{x_k\}}, T\delta_{\{y_k\}} \rangle_{L^2(\mathbb{R}^n)} - \langle \delta_{\{x_k\}}, T\delta_{\{y\}} \rangle_{L^2(\mathbb{R}^n)}\right| \\ &\qquad + \left|\langle \delta_{\{x_k\}}, T\delta_{\{y\}} \rangle_{L^2(\mathbb{R}^n)} - \langle \delta_{\{x\}}, T\delta_{\{y\}} \rangle_{L^2(\mathbb{R}^n)}\right| \\ &\quad \leqslant \left|\delta_{\{x_k\}}\right|_{-\frac{n}{2}-\varepsilon} \left|T\delta_{\{y_k\}} - T\delta_{\{y\}}\right|_{\frac{n}{2}+\varepsilon} + \left|T\delta_{\{y\}}(x_k) - T\delta_{\{y\}}(x)\right| \\ &\quad \leqslant M \|T\| \left|\delta_{\{y\}} - \delta_{\{y_k\}}\right|_{-\frac{n}{2}-\varepsilon} + \eta \leqslant (M\|T\| + 1)\,\eta.\end{aligned}$$

Next, we turn to the second part of the theorem. Since $H^{\frac{n}{2}+\varepsilon+1}(\mathbb{R}^n) \hookrightarrow H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n)$, the map $t$ is continuous by the first part of the theorem. To prove differentiability, it suffices to observe that $t$ has continuous (weak) partial derivatives. Since $T\partial_j \in \mathcal{B}\big(H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n), H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n)\big)$ and $\partial_j T \in \mathcal{B}\big(H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n), H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n)\big)$ for all $j \in \{1, \ldots, n\}$, the assertion also follows from the first part as one observes that

$$\begin{aligned}(\partial_j t)(x, y) &= \left\langle (\partial_j \delta)_{\{x\}}, T\delta_{\{y\}} \right\rangle_{L^2(\mathbb{R}^n)} = -\left\langle \delta_{\{x\}}, \partial_j T\delta_{\{y\}} \right\rangle_{L^2(\mathbb{R}^n)}, \\ (\partial_{j+n} t)(x, y) &= \langle \delta_{\{x\}}, T\partial_j \delta_{\{y\}} \rangle, \quad j \in \{1, \ldots, n\},\ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.\end{aligned}$$ □

In the applications discussed later on, we shall be confronted with operators being already defined (or being extendable) to the whole fractional Sobolev scale. So the standard situation in which we will apply Theorem 5.1 is summarized in the following corollary, with some examples in the succeeding proposition.

**Corollary 5.3.** *Let $n \in \mathbb{N}$, $k > n$, $S, T \colon \bigcup_{\ell\in\mathbb{Z}} H^\ell(\mathbb{R}^n) \to \bigcup_{\ell\in\mathbb{Z}} H^\ell(\mathbb{R}^n)$. Assume that for all $\ell \in \mathbb{R}$, $T \in \mathcal{B}\big(H^\ell(\mathbb{R}^n), H^{\ell+k}(\mathbb{R}^n)\big)$ and $S \in \mathcal{B}\big(H^\ell(\mathbb{R}^n), H^{\ell+k+1}(\mathbb{R}^n)\big)$, and introduce the maps*

$$\begin{aligned}t &\colon \mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto \left\langle \delta_{\{x\}}, T\delta_{\{y\}} \right\rangle_{L^2(\mathbb{R}^n)}, \\ s &\colon \mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto \left\langle \delta_{\{x\}}, S\delta_{\{y\}} \right\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

*Then $t$ is bounded and continuous, and the map $s$ is bounded and continuously differentiable with bounded derivatives.* (*See* (5.1) *and* (5.3) *for* $H^s(\mathbb{R}^n)$ *and* $\delta_{\{x\}}$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}$.)

*Proof.* This is a direct consequence of Theorem 5.1. □

**Proposition 5.4.** *Let* $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) > 0$, $\ell \in \mathbb{R}$, *and* $\Phi \in C_b^\infty(\mathbb{R}^n)$. *Then* $R_\mu = (-\Delta + \mu)^{-1} \in \mathcal{B}\big(H^\ell(\mathbb{R}^n), H^{\ell+2}(\mathbb{R}^n)\big)$ *and*[1] $\Phi \in \mathcal{B}\big(H^\ell(\mathbb{R}^n)\big)$.

*Proof.* For $\ell \in \mathbb{Z}$, the first assertion follows easily with the help of the Fourier transform, the second assertion is a straightforward induction argument for $\ell \in \mathbb{N}_0$; for $\ell \in -\mathbb{N}$ the result follows by duality. The results for $\ell \in \mathbb{R}$ follow by interpolation, see [96, Theorem 2.4.2]. □

The main issue of the considerations in this section are estimates of continuous integral kernels on the respective diagonals. An elementary estimate can be shown for integral operators which are induced by commutators with multiplication operators as the following result confirms.

**Proposition 5.5.** *Let* $n \in \mathbb{N}$, $\varepsilon > 0$, $m \in \mathbb{N}$, *and assume that* $T \colon H^{-(n/2)-\varepsilon}(\mathbb{R}^n) \to H^{(n/2)+\varepsilon}(\mathbb{R}^n)$ *is linear and continuous, and that* $\Phi \in C_b^\infty(\mathbb{R}^n)$. *Then the map*

$$t_\Phi \colon \mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto \big\langle \delta_{\{x\}}, [\Phi, T]\, \delta_{\{y\}} \big\rangle_{L^2(\mathbb{R}^n)} \in \mathbb{C},$$

*where* $[\Phi, T]$ *is given by* (2.2), *is well-defined, continuous, bounded, and satisfies* $t_\Phi(x,x) = 0$, $x \in \mathbb{R}^n$.

*Proof.* By Proposition 5.4 and Theorem 5.1, one gets that $t_\Phi$ is well-defined, continuous, and bounded. For $x \in \mathbb{R}^n$ one then computes

$$\begin{aligned} \big\langle \delta_{\{x\}}, [\Phi, T]\, \delta_{\{y\}} \big\rangle &= \big\langle \delta_{\{x\}}, (\Phi T - T\Phi)\, \delta_{\{x\}} \big\rangle \\ &= \big\langle \delta_{\{x\}}, \Phi T \delta_{\{x\}} \big\rangle - \big\langle \delta_{\{x\}}, (T\Phi)\, \delta_{\{x\}} \big\rangle \\ &= \big\langle \Phi^* \delta_{\{x\}}, T\delta_{\{x\}} \big\rangle - \big\langle \delta_{\{x\}}, T\, (\Phi\delta)_{\{x\}} \big\rangle \\ &= \big\langle \overline{\Phi(x)} \delta_{\{x\}}, T\delta_{\{x\}} \big\rangle - \big\langle \delta_{\{x\}}, T\Phi(x)\delta_{\{x\}} \big\rangle \\ &= \big\langle \delta_{\{x\}}, \Phi(x) T\delta_{\{x\}} \big\rangle - \big\langle \delta_{\{x\}}, T\Phi(x)\delta_{\{x\}} \big\rangle = 0. \end{aligned}$$ □

The next lemma also discusses properties of the integral kernel of a commutator, however, in the following situation, we shall address the commutator with differentiation.

**Lemma 5.6.** *Let* $T \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big)$ *be induced by the continuously differentiable integral kernel* $t \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $j \in \{1, \dots, n\}$. *If* $[\partial_j, T] \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big)$, *then* $[\partial_j, T]$ *defined by* (2.2) *is an operator induced by the integral kernel* $\partial_j t + \partial_{j+n} t$.

*Proof.* Let $x \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. One computes

$$\begin{aligned} ([\partial_j, T]f)(x) &= (\partial_j T f)(x) - (T\partial_j f)(x) \\ &= \partial_j \int_{\mathbb{R}^n} t(x,y) f(y)\, d^n y - \int_{\mathbb{R}^n} t(x,y)(\partial_j f)(y)\, d^n y \\ &= \int_{\mathbb{R}^n} (\partial_j t)(x,y) f(y)\, d^n y + \int_{\mathbb{R}^n} (\partial_{j+n} t)(x,y) f(y)\, d^n y, \end{aligned}$$

[1]We recall Remark 2.1: The symbol $\Phi$ is interpreted as the operator of multiplication by the function $\Phi$. If $\ell < 0$, this should read as multiplication in the distributional sense.

using an integration by parts to arrive at the last equality. □

*Remark* 5.7. We elaborate on an important consequence of Lemma 5.6 as follows: For $j \in \{1, \dots, n\}$, let $T_j \in \mathcal{B}(L^2(\mathbb{R}^n))$ be induced by the continuously differentiable integral kernel $t_j \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. Assume that $[\partial_j, T_j] \in \mathcal{B}(L^2(\mathbb{R}^n))$, $j \in \{1, \dots, n\}$, and consider the operator

$$T \coloneqq \sum_{j=1}^{n} [\partial_j, T_j].$$

By Lemma 5.6 one infers that the integral kernel $t$ for $T$ may be computed as follows,

$$t(x,y) = \sum_{j=1}^{n} (\partial_j t_j + \partial_{j+n} t_j)(x,y), \quad x, y \in \mathbb{R}^n.$$

Moreover, for $g \coloneqq \{x \mapsto t_j(x,x)\}_{j \in \{1,\dots,n\}}$,

$$\begin{aligned} t(x,x) &= \sum_{j=1}^{n} (\partial_j (y \mapsto t_j(y,y)))(x) \\ &= \operatorname{div}(g(x)), \quad x \in \mathbb{R}^n. \end{aligned} \tag{5.5}$$

This observation will turn out to be useful when computing the trace of certain operators. ⋄

The remaining section is devoted to obtaining pointwise estimates of various integral operators on the diagonal. For convenience, we recall the Γ-function (cf. [1, Sect. 6.1]), given by

$$\Gamma(z) \coloneqq \int_0^\infty t^{z-1} e^{-t}\, dt, \quad z \in \mathbb{C}_{\mathrm{Re}>0},$$

as well as the $n-1$-dimensional volume of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$,

$$\omega_{n-1} = \frac{2\pi^{(n/2)}}{\Gamma\big(n/2\big)}. \tag{5.6}$$

**Proposition 5.8.** *Let $n, m \in \mathbb{N}$, $m > (n+1)/2$, $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) > 0$, and $R_\mu$, $\delta_{\{0\}}$, and $Q$ be given by* (4.6), (5.3), *and* (4.1), *respectively. Then*

$$\big|R_\mu^m \delta_{\{0\}}(0)\big| \leqslant \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \left(\sqrt{\mathrm{Re}(\mu)}\right)^n c, \tag{5.7}$$

$$\big|Q R_\mu^m \delta_{\{0\}}(0)\big| \leqslant \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \left(\sqrt{\mathrm{Re}(\mu)}\right)^{n+1} c', \tag{5.8}$$

*with*

$$c = (2\pi)^{-n}\, \omega_{n-1} \int_0^\infty r^{n-1} \big[r^2+1\big]^{-(n+3)/2} dr, \tag{5.9}$$

*and*

$$c' = (2\pi)^{-n} \sqrt{n}\omega_{n-1} \int_0^\infty r^{n} \big[r^2+1\big]^{-(n+3)/2} dr, \tag{5.10}$$

*where $\omega_{n-1}$ is given by* (5.6).

*Proof.* We estimate $R_\mu^m \delta_{\{0\}}(0)$ with the help of the Fourier transform as follows.

$$\begin{aligned}
\big|R_\mu^m \delta_{\{0\}}(0)\big| &= \big|\big\langle R_\mu^m \delta_{\{0\}}, \delta_{\{0\}}\big\rangle\big| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} \frac{1}{\big(|\xi|^2 + \mu\big)^m} \, d^n\xi \right| \\
&\leqslant (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{1}{\big(|\xi|^2 + \mathrm{Re}(\mu)\big)^m} \, d^n\xi \\
&= (2\pi)^{-n} \, \omega_{n-1} \int_0^\infty \frac{r^{n-1}}{(r^2 + \mathrm{Re}(\mu))^m} \, dr \\
&= (2\pi)^{-n} \, \omega_{n-1} \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \int_0^\infty \frac{r^{n-1}}{\left(\left(\frac{r}{\sqrt{\mathrm{Re}(\mu)}}\right)^2 + 1\right)^m} \, dr \\
&= (2\pi)^{-n} \, \omega_{n-1} \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \int_0^\infty \frac{\big(t\sqrt{\mathrm{Re}(\mu)}\big)^{n-1}}{(t^2+1)^m} \sqrt{\mathrm{Re}(\mu)} \, dt \\
&= (2\pi)^{-n} \, \omega_{n-1} \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \big(\sqrt{\mathrm{Re}(\mu)}\big)^n \int_0^\infty \frac{t^{n-1}}{(t^2+1)^m} \, dt.
\end{aligned}$$

In a similar fashion, one estimates (5.8), however, first we recall from (4.5) that $\left|\sum_{j=1}^n \gamma_{j,n}(-i)\xi_j\right| \leqslant \sqrt{n}\,|\xi|$, $\xi \in \mathbb{R}^n$. Hence, one arrives at

$$\begin{aligned}
\big|Q R_\mu^m \delta_{\{0\}}(0)\big| &\leqslant (2\pi)^{-n} \sqrt{n} \int_{\mathbb{R}^n} \frac{|\xi|}{\big(|\xi|^2 + \mathrm{Re}(\mu)\big)^m} \, d^n\xi \\
&= (2\pi)^{-n} \sqrt{n} \omega_{n-1} \int_0^\infty \frac{r^n}{(r^2 + \mathrm{Re}(\mu))^m} \, dr \\
&= (2\pi)^{-n} \sqrt{n} \omega_{n-1} \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \int_0^\infty \frac{r^n}{\left(\left(\frac{r}{\sqrt{\mathrm{Re}(\mu)}}\right)^2 + 1\right)^m} \, dr \\
&= (2\pi)^{-n} \sqrt{n} \omega_{n-1} \left(\frac{1}{\mathrm{Re}(\mu)}\right)^m \big(\sqrt{\mathrm{Re}(\mu)}\big)^{n+1} \int_0^\infty \frac{t^n}{(t^2+1)^m} \, dt.
\end{aligned}$$

□

The main observation in this subsection, Lemma 5.14, needs some preparations which deal with the fundamental solution of the Helmholtz equation on $\mathbb{R}^n$ for $n \geqslant 3$ odd, to be introduced in (5.11).

**Lemma 5.9.** *Let $n, N \in \mathbb{N}$, and $x_1, \dots, x_N \in \mathbb{R}^n$. Then*

$$|x_1| + \sum_{j=1}^{N-1} |x_{j+1} - x_j| \geqslant \max_{1 \leqslant k \leqslant N} |x_k| \, .$$

*Proof.* We proceed by induction. The case $N = 1$ is clear. For $N \in \mathbb{N}$, one has

$$|x_1| + \sum_{j=1}^{N} |x_{j+1} - x_j| \geqslant |x_1| + \sum_{j=1}^{N} \big[\, |x_{j+1}| - |x_j| \,\big] = |x_{N+1}| \, .$$

Thus, employing the induction hypothesis, one gets that

$$|x_1| + \sum_{j=1}^{N} |x_{j+1} - x_j| \geqslant \max_{1 \leqslant k \leqslant N} |x_k| \vee |x_{N+1}| = \max_{1 \leqslant \ell \leqslant N+1} |x_\ell| \, .$$

□

**Lemma 5.10.** *Let $\alpha > 0$, $\beta > 0$. Then the map*

$$\phi\colon [0,\infty) \ni r \mapsto \left(1+\frac{1}{2}r\right)^{\alpha} e^{-\beta r}$$

*satisfies*

$$|\phi(r)| \leqslant \begin{cases} [\alpha/(2\beta)]^{\alpha} e^{-\alpha+2\beta}, & \alpha > 2\beta, \\ 1, & \alpha \leqslant 2\beta, \end{cases} \quad r \geqslant 0.$$

*Proof.* From

$$\begin{aligned} \phi'(r) &= \left(\frac{1}{2}\alpha\left(1+\frac{1}{2}r\right)^{\alpha-1} - \beta\left(1+\frac{1}{2}r\right)^{\alpha}\right)e^{-\beta r} \\ &= \left(\frac{1}{2}\alpha - \beta\left(1+\frac{1}{2}r\right)\right)\left(1+\frac{1}{2}r\right)^{\alpha-1} e^{-\beta r} \end{aligned}$$

one gets with $r^* := (\alpha/\beta) - 2$ that $\phi'(r^*) = 0$ if $r^* > 0$. Thus, by $\phi(0) = 1$ and $\phi(r) \to 0$ as $r \to \infty$, one obtains the assertion. □

Next, we shall concentrate on pointwise estimates for the fundamental solution of the Helmholtz equation. We denote the integral kernel (i.e., the Helmholtz Green's function) associated with $(-\Delta - z)^{-1}$ by $E_n(z;x,y)$, $x,y \in \mathbb{R}^n$, $x \neq y$, $n \in \mathbb{N}$, $n \geqslant 2$, $z \in \mathbb{C}$. Then,

$$\begin{aligned} &E_n(z;x,y) \\ &\quad = \begin{cases} (i/4)\big(2\pi z^{-1/2}|x-y|\big)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x-y|\big), & n \geqslant 2,\ z \in \mathbb{C}\backslash\{0\}, \\ -(2\pi)^{-1}\ln(|x-y|), & n = 2,\ z = 0, \\ [(n-2)\omega_{n-1}]^{-1}|x-y|^{2-n}, & n \geqslant 3,\ z = 0, \end{cases} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Im}\big(z^{1/2}\big) \geqslant 0,\ x,y \in \mathbb{R}^n,\ x \neq y, \end{aligned} \tag{5.11}$$

where $H^{(1)}_{\nu}(\cdot)$ denotes the Hankel function of the first kind with index $\nu \geqslant 0$ (cf. [1, Sect. 9.1]), and $\omega_{n-1}$ is given by (5.6). We will directly work with the explicit formula (5.11), even though one could also employ the Laplace transform connection between the resolvent and the semigroup of $-\Delta$ which manifests itself in the formula,

$$E_n(z;x,y) = \int_{[0,\infty)} (4\pi t)^{-n/2} e^{-|x-y|^2/(4t)} e^{zt}\, dt, \quad \operatorname{Re}(z) < 0,\ x,y \in \mathbb{R}^n,\ x \neq y. \tag{5.12}$$

Later on, we need the following reformulation of the Helmholtz Green's function in odd space dimensions. We will use an explicit expression for the Hankel function of the first kind.

**Lemma 5.11.** *Let $n = 2\widehat{n}+1 \in \mathbb{N}_{\geqslant 3}$ odd, $\mu \in \mathbb{C}_{\mathrm{Re}>0}$. We denote*

$$\mathcal{E}_n(-z,r) := E_n(z;x,y), \quad r > 0,\ z \in \mathbb{C}\backslash\{0\},$$

*where $x,y \in \mathbb{R}^n$ are such that $|x-y| = r$. Then the following formula holds*

$$\mathcal{E}_n(\mu,r) = \left(\frac{\sqrt{\mu}}{2}\right)^{\widehat{n}-1} (2\pi r)^{-\widehat{n}}\, e^{-\sqrt{\mu}r} \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!}\left(\frac{1}{2\sqrt{\mu}r}\right)^{k}.$$

*Proof.* Our branch of $\sqrt{\cdot}$ is chosen such that $\sqrt{-z} = i\sqrt{z}$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. We use the following representation of the Hankel function of the first kind taken from [57, 8.466.1],

$$H^{(1)}_{\widehat{n}-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{-\widehat{n}} e^{iz} \sum_{k=0}^{\widehat{n}-1} (-1)^k \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \frac{1}{(2iz)^k}, \quad \widehat{n} \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
\mathcal{E}_n(\mu, r) &= \frac{i}{4} \bigg(\frac{2\pi r}{i\mu^{1/2}}\bigg)^{\frac{1}{2}-\widehat{n}} H^{(1)}_{\widehat{n}-\frac{1}{2}}\big(i\mu^{1/2} r\big) \\
&= \frac{i}{4} \bigg(\frac{2\pi r}{i\mu^{1/2}}\bigg)^{\frac{1}{2}-\widehat{n}} \sqrt{\frac{2}{\pi i \mu^{1/2} r}} i^{-\widehat{n}} e^{ii\mu^{1/2} r} \sum_{k=0}^{\widehat{n}-1} (-1)^k \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \frac{1}{(2ii\mu^{1/2} r)^k} \\
&= \frac{i}{4} \bigg(\frac{2\pi r}{i\mu^{1/2}}\bigg)^{\frac{1}{2}-\widehat{n}} \sqrt{\frac{2}{\pi i \mu^{1/2} r}} i^{-\widehat{n}} e^{-\mu^{1/2} r} \sum_{k=0}^{\widehat{n}-1} (-1)^k \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \frac{1}{(-2\mu^{1/2} r)^k} \\
&= \bigg(\frac{\sqrt{\mu}}{2}\bigg)^{\widehat{n}-1} (2\pi r)^{-\widehat{n}} \, e^{-\sqrt{\mu} r} \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \bigg(\frac{1}{2\sqrt{\mu} r}\bigg)^k. \qquad \square
\end{aligned}$$

As a first corollary to be drawn from the explicit formula in the latter result, we now derive some estimates of the Helmholtz Green's function.

**Lemma 5.12.** *Let $n = 2\widehat{n}+1 \in \mathbb{N}_{\geqslant 3}$ odd, $\mu \in \mathbb{C}_{\mathrm{Re}>0}$. We denote*

$$\mathcal{E}_n(\mu, r) \coloneqq E_n(-\mu; x, y), \quad r > 0, \tag{5.13}$$

*with $x, y \in \mathbb{R}^n$ such that $|x-y| = r$. Then the following assertions (i)–(iii) hold:*
*(i) Assume that $\mu > 0$, then for all $r > 0$,*

$$\mathcal{E}_n(\mu, r) > 0. \tag{5.14}$$

*(ii) For all $r > 0$,*

$$|\mathcal{E}_n(\mu, r)| \leqslant \Big(\sqrt{\cos(\arg(\mu))}\Big)^{1-\widehat{n}} \mathcal{E}_n(\mathrm{Re}(\mu), r). \tag{5.15}$$

*(iii) Assume that $\mu > 0$, then for all $r > 0$,*

$$\exp\big(\sqrt{\mu} r/2\big) \mathcal{E}_n(\mu, r) \leqslant 2^{\widehat{n}-1} \mathcal{E}_n(\mu/4, r). \tag{5.16}$$

*Proof.* Assertion (5.14) is clear due to the fact that $\mathcal{E}_n$ is the fundamental solution of the positive, self-adjoint operator $(-\Delta + \mu)$. (Alternatively, one can also use the explicit representation of $\mathcal{E}_n(\cdot,\cdot)$ in Lemma 5.11.)

In view of Lemma 5.11, in order to prove (5.15), it suffices to prove the following two facts,

$$\frac{|\sqrt{\mu}|}{\sqrt{\mathrm{Re}(\mu)}} = \frac{1}{\sqrt{\cos(\arg(\mu))}} \text{ and } |\mathrm{Re}(\sqrt{\mu})| \geqslant \sqrt{\mathrm{Re}(\mu)}. \tag{5.17}$$

To show these assertions, let $\varrho > 0$ and $\vartheta \in (-\pi/2, \pi/2)$ such that $\mu = \varrho e^{i\vartheta}$. Then $\sqrt{\mu} = \sqrt{\varrho} e^{i\vartheta/2} = \sqrt{\varrho}\cos(\vartheta/2) + i\sqrt{\varrho}\sin(\vartheta/2)$ as well as $\sqrt{\mathrm{Re}(\mu)} = \sqrt{\varrho}\sqrt{\cos(\vartheta)}$. From

$$\sqrt{\cos(\vartheta)} = \sqrt{\big(\cos(\vartheta/2)\big)^2 - \big(\sin(\vartheta/2)\big)^2} \leqslant \sqrt{\big(\cos(\vartheta/2)\big)^2} = \cos(\vartheta/2),$$

and

$$\frac{|\sqrt{\mu}|}{\sqrt{\operatorname{Re}(\mu)}} = \frac{|\sqrt{\varrho} e^{i\vartheta/2}|}{\sqrt{\varrho}\sqrt{\cos(\arg(\mu))}},$$

assertion (5.17) follows.

Finally, we turn to the proof of (5.16). Given the representation of $\mathcal{E}_n(\cdot,\cdot)$ in Lemma 5.11, one concludes that

$$\begin{aligned}
\exp\left(\frac{\sqrt{\mu}}{2} r\right) \mathcal{E}_n(\mu, r) &= \exp\left(\frac{\sqrt{\mu}}{2} r\right) \left(\frac{\sqrt{\mu}}{2}\right)^{\widehat{n}-1} (2\pi r)^{-\widehat{n}}\, e^{-\sqrt{\mu} r} \\
&\quad \times \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \left(\frac{1}{2\sqrt{\mu} r}\right)^k \\
&= \left(\frac{\sqrt{\mu}}{2}\right)^{\widehat{n}-1} (2\pi r)^{-\widehat{n}}\, e^{-\frac{\sqrt{\mu}}{2} r} \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \left(\frac{1}{2\sqrt{\mu} r}\right)^k \\
&\leqslant 2^{\widehat{n}-1} \left(\frac{\sqrt{\mu/4}}{2}\right)^{\widehat{n}-1} (2\pi r)^{-\widehat{n}}\, e^{-\sqrt{\frac{\mu}{4}} r} \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \left(\frac{1}{2\sqrt{\frac{\mu}{4}} r}\right)^k. \qquad \square
\end{aligned}$$

Next, we obtain similar results for the derivative of the fundamental solution.

**Lemma 5.13.** *Let $n = 2\widehat{n}+1 \in \mathbb{N}_{\geqslant 3}$ odd. Then, for all $\mu \in \mathbb{C}_{\operatorname{Re}>0}$, there exists $q_\mu \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $q_\mu(|\cdot|) \in L^1(\mathbb{R}^n)$, such that the following properties (i)–(iii) hold:*

*(i) For all $j \in \{1, \dots, n\}$ and $\mu \in \mathbb{C}$, $\operatorname{Re}(\mu) > 0$, and for all $x, y \in \mathbb{R}^n$, $x \neq y$,*

$$|\partial_j (\xi \mapsto E_n(-\mu; \xi, y))(x)| \leqslant q_\mu(|x-y|). \tag{5.18}$$

*(ii) For all $\mu \in \mathbb{C}$, $\operatorname{Re}(\mu) > 0$,*

$$q_\mu(r) \leqslant \left(1/\sqrt{\cos(\arg(\mu))}\right)^{\widehat{n}} q_{\operatorname{Re}(\mu)}(r), \quad r > 0. \tag{5.19}$$

*(iii) For all $\mu > 0$,*

$$\exp(\sqrt{\mu} r/2)\, q_\mu(r) \leqslant 2^{\widehat{n}} q_{\mu/4}(r), \quad r > 0. \tag{5.20}$$

*Proof.* For $r > 0$, with $\mathcal{E}_n(\mu, r)$ as in Lemma 5.12 (and with the help of Lemma 5.11), one obtains the following derivative of $\mathcal{E}_n(\cdot,\cdot)$ with respect to the second variable,

$$\begin{aligned}
(\partial_r \mathcal{E}_n)(\mu, r) = &-\left(\frac{\sqrt{\mu}}{2}\right)^{\widehat{n}-1} (2\pi)^{-\widehat{n}} e^{-\sqrt{\mu} r} \\
&\times \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \left(\frac{1}{2\sqrt{\mu}}\right)^k r^{-\widehat{n}-k-1} \left(\sqrt{\mu} r + (\widehat{n}+k)\right).
\end{aligned}$$

Define for $r > 0$,

$$\begin{aligned}
q_\mu(r) := \Bigg| &\left(\frac{\sqrt{\mu}}{2}\right)^{\widehat{n}-1} (2\pi)^{-\widehat{n}} e^{-\sqrt{\mu} r} \\
&\times \sum_{k=0}^{\widehat{n}-1} \frac{(\widehat{n}+k-1)!}{k!(\widehat{n}-k-1)!} \left(\frac{1}{2\sqrt{\mu}}\right)^k r^{-\widehat{n}-k-1} \left(\sqrt{\mu} r + (\widehat{n}+k)\right) \Bigg|.
\end{aligned} \tag{5.21}$$

Then $q_\mu(|\cdot|) \in L^1(\mathbb{R}^n)$. Indeed, due to the presence of the $e^{-\sqrt{\mu}r}$-term, only integrability at $x = 0$ is an issue here. Since the order of the singularity of $q_\mu(|x|)$ at $x = 0$ is at most $|x|^{-n}$ since $\widehat{n} + (\widehat{n} - 1) + 1 = 2\widehat{n} < 2\widehat{n} + 1 = n$, also integrability of $q_\mu(|\cdot|)$ at $x = 0$ is ensured.

To prove (5.18), one observes that for fixed $y \in \mathbb{R}^n$ and $y \neq x \in \mathbb{R}^n$,

$$|\partial_j \left(\xi \mapsto \mathcal{E}_n(\mu, |\xi - y|)\right)(x)| = q_\mu(|x-y|) \left|\frac{x_j - y_j}{|x-y|}\right| \leqslant q_\mu(|x-y|).$$

The assertion in (5.19) follows analogously to that of (5.15) with an explicit representation of $\sqrt{\mu}$, together with the observation that $1/\sqrt{\cos(\arg(\mu))} \geqslant 1$. The same arguments apply to the proof of (5.20). $\square$

Having established the preparations for estimating the integral kernel of products of resolvents $R_\mu$ of the Laplace operator and a multiplication operator $\Psi_j$, we finally come to the fundamental estimates (5.23) in Lemma 5.14. All the following results will be of a similar type. Namely, consider a product

$$\Psi_1 R_\mu \Psi_2 R_\mu \cdots \Psi_m R_\mu, \tag{5.22}$$

of smooth, bounded functions $\Psi_j$, $j \in \{1, \dots, m\}$, identified as multiplication operators in $L^2(\mathbb{R}^n)$ and $R_\mu = (-\Delta + \mu)^{-1}$ for some $\mu \in \mathbb{C}_{\mathrm{Re}>0}$. If $m$ is sufficiently large (depending on the space dimension $n$), the operator introduced in (5.22) has a continuous integral kernel $t\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. Roughly speaking, we will show that the behavior of $x \mapsto t(x,x)$ is determined by the (algebraic) decay properties of all the functions $\Psi_j$, $j \in \{1, \dots, n\}$, that is, if $\Psi_j$ decays like $|x|^{-\alpha_j}$ for large $|x|$ for some $\alpha_j \geqslant 0$, $j \in \{1, \dots, m\}$, then $x \mapsto t(x,x)$ decays as $|x|^{-\sum_{j=1}^m \alpha_j}$. We will, however, need a more precise estimate. Namely, we also need to establish at the same time the overall constant of this decay behavior as a function of $\mu$. That is why we needed to establish Proposition 5.8, see also Remark 5.15 below.

The precise statement regarding the estimate of the diagonal of such a continuous integral kernel reads as follows:

**Lemma 5.14.** *Let $n = 2\widehat{n} + 1 \in \mathbb{N}_{\geqslant 3}$, $m \geqslant \widehat{n} + 1$, and assume that $\Psi_1, \dots, \Psi_{m+1} \in C_b^\infty(\mathbb{R}^n)$, and $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) > 0$. Assume that there exists $\alpha_j, \kappa_j \in \mathbb{R}_{\geqslant 0}$, $j \in \{1, \dots, m+1\}$, such that*

$$|\Psi_j(x)| \leqslant \kappa_j (1 + |x|)^{-\alpha_j}, \quad x \in \mathbb{R}^n,\ j \in \{1, \dots, m+1\}.$$

*Consider the integral kernels $t$ and $t_k$ of $T := \prod_{j=1}^m \Psi_j R_\mu$ and $T_k := \prod_{j=1}^{m+1} \Psi_{j,k} R_\mu$, respectively* (*cf.* Remark 2.1), *where $\Psi_{j,k} = \Psi_j(1 - \delta_{jk}) + \delta_{jk}\Psi_j Q$, $k \in \{1, \dots, n\}$, with $R_\mu$ and $Q$ given by* (4.1) *and* (4.6), *respectively. Then $t$ and $t_k$ are continuous and there exists $\kappa' > 0$ such that*

$$|t(x,x)| \leqslant \kappa' [1 + (|x|/2)]^{-\sum_{j=1}^m \alpha_j}, \quad |t_k(x,x)| \leqslant \kappa' [1 + (|x|/2)]^{-\sum_{j=1}^{m+1} \alpha_j}, \quad x \in \mathbb{R}^n. \tag{5.23}$$

*For $\sqrt{\mathrm{Re}(\mu)} > 2\sum_{j=1}^{m+1} \alpha_j$, one can choose, with $c_{\arg(\mu)} := \cos(\arg(\mu))^{-1/2}$,*

$$\kappa' = \left[(2c_{\arg(\mu)})^{\widehat{n}-1}\right]^m \kappa_1 \cdots \kappa_m \left(\frac{4}{\mathrm{Re}(\mu)}\right)^m \left(\sqrt{\frac{\mathrm{Re}(\mu)}{4}}\right)^n c$$

*in the first estimate in* (5.23), *and*

$$\kappa' = c_{\arg(\mu)} \left[(c_{\arg(\mu)})^{\widehat{n}-1}\right]^m (2^{\widehat{n}})^m \kappa_1 \cdots \kappa_{m+1} \left(\frac{4}{\mathrm{Re}(\mu)}\right)^{m+1} \left(\sqrt{\frac{\mathrm{Re}(\mu)}{4}}\right)^{n+1} c'$$

*in the second, with* $c$ *and* $c'$ *given by* (5.9) *and* (5.10)*, respectively.*

*Proof.* We shall only prove the assertion for $T$. The other assertions follow from the fact that the integral kernel of $QR_\mu$ can be bounded by $x \mapsto c^{\widehat{n}}_{\arg(\mu)}\mu q_{\mathrm{Re}(\mu)}(|x|)$, see Lemma 5.13. Moreover, we shall exploit that the exponential estimates (5.16) and (5.19) in Lemmas 5.12 and 5.13, respectively, are essentially the same.

The stated continuity of the integral kernels follows from $2m \geqslant 2\widehat{n} + 2 > n$, Corollary 5.3, and Proposition 5.4. Indeed, Proposition 5.4 implies that

$$T \in L\left(H^{\ell}\left(\mathbb{R}^n\right), H^{\ell+2m}\left(\mathbb{R}^n\right)\right), \quad \ell \in \mathbb{R}.$$

Thus, by Corollary 5.3,

$$t\colon \mathbb{R}^n \times \mathbb{R}^n \colon (x,y) \mapsto \left\langle \delta_{\{x\}}, T\delta_{\{y\}} \right\rangle$$

is continuous as $2m > n$, with $\delta_{\{x\}}$, $x \in \mathbb{R}^n$, defined in (5.3).

We denote the integral kernel of $R_\mu = (-\Delta + \mu)^{-1}$ by $r_\mu$. Then one notes that

$$r_\mu(x-y) = E_n(-\mu; x, y) = \mathcal{E}_n(\mu; |x-y|).$$

For simplicity, we now assume that $\mu$ is real (one recalls the estimate $|r_\mu| \leqslant c_{\arg(\mu)} r_{\mathrm{Re}(\mu)}$ with a positive real number $c_{\arg(\mu)}$ depending on $\arg(\mu)$, see (5.15)). One observes that

$$\frac{1}{1+|x_1+x|} \leqslant \frac{1}{1+||x_1|-|x||} = \frac{1}{1+|x|-|x_1|} \leqslant \frac{1}{1+\frac{1}{2}|x|}, \qquad x, x_1 \in \mathbb{R}^n,\ |x| \geqslant 2|x_1|.$$

On the other hand, one obviousy also has

$$\frac{1}{1+|x_1+x|} \leqslant 1, \quad |x| \leqslant 2|x_1|.$$

Introducing the sets,

$$\begin{aligned} B(R) &\coloneqq \left\{(x_1, \ldots . x_{m-1}) \in (\mathbb{R}^n)^{m-1} \;\middle|\; \max_{1\leqslant j\leqslant m-1} |x_j| \leqslant R\right\}, \\ \complement B(R) &\coloneqq (\mathbb{R}^n)^{m-1} \backslash B(R), \quad R \geqslant 0, \end{aligned}$$

one computes for $x \in \mathbb{R}^n$, with $\widetilde{\kappa} \coloneqq \kappa_1 \cdots \kappa_{m+1}$,

$$\begin{aligned} |t(x,x)| &= \left| \Psi_1(x) \int_{(\mathbb{R}^n)^{m-1}} r_\mu(x-x_1)\Psi_2(x_1)r_\mu(x_1-x_2)\cdots\Psi_m(x_{m-1}) \right. \\ &\qquad\qquad \left. \times r_\mu(x_{m-1}-x)\, d^n x_1 \cdots d^n x_{m-1} \right| \\ &\leqslant \int_{(\mathbb{R}^n)^{m-1}} |\Psi_1(x)|\, r_\mu(x-x_1) \cdots |\Psi_m(x_{m-1})|\, r_\mu(x_{m-1}-x) \\ &\qquad \times d^n x_1 \cdots d^n x_{m-1} \\ &= \int_{(\mathbb{R}^n)^{m-1}} |\Psi_1(x)|\, r_\mu(x_1) \cdots |\Psi_m(x_{m-1}+x)|\, r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} \\ &\leqslant \widetilde{\kappa} \int_{(\mathbb{R}^n)^{m-1}} \left(\frac{1}{1+|x|}\right)^{\alpha_1} r_\mu(x_1) \cdots \left(\frac{1}{1+|x_{m-1}+x|}\right)^{\alpha_m} \\ &\qquad \times r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} \\ &= \widetilde{\kappa} \int_{B(|x|/2)} \left(\frac{1}{1+|x|}\right)^{\alpha_1} r_\mu(x_1) \cdots \left(\frac{1}{1+|x_{m-1}+x|}\right)^{\alpha_m} \end{aligned}$$

$$
\begin{aligned}
&\quad \times r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} \\
&+ \widetilde{\kappa} \int_{\complement B(|x|/2)} \bigg(\frac{1}{1+|x|}\bigg)^{\alpha_1} r_\mu(x_1) \cdots \bigg(\frac{1}{1+|x_{m-1}+x|}\bigg)^{\alpha_m} \\
&\qquad \times r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} \\
&\leqslant \frac{\widetilde{\kappa}}{\big(1+\frac{1}{2}|x|\big)^{\sum_{j=1}^m \alpha_j}} \int_{B(|x|/2)} r_\mu(x_1) r_\mu(x_1 - x_2) \cdots r_\mu(x_{m-1}) \\
&\qquad \times d^n x_1 \cdots d^n x_{m-1} \\
&+ \widetilde{\kappa} \int_{\complement B(|x|/2)} r_\mu(x_1) r_\mu(x_1 - x_2) \cdots r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} \\
&\leqslant \frac{\widetilde{\kappa}}{\big(1+\frac{1}{2}|x|\big)^{\sum_{j=1}^m \alpha_j}} \int_{(\mathbb{R}^n)^{m-1}} r_\mu(x_1) r_\mu(x_1 - x_2) \cdots r_\mu(x_{m-1}) \\
&\qquad \times d^n x_1 \cdots d^n x_{m-1} \\
&+ \widetilde{\kappa} \int_{\complement B(|x|/2)} r_\mu(x_1) r_\mu(x_1 - x_2) \cdots r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1}.
\end{aligned}
$$

By Lemma 5.9 one recalls that for $x_1, \dots, x_{m-1} \in \mathbb{R}^n$,

$$
|x_1| + \sum_{j=1}^{m-2} |x_{j+1} - x_j| + |x_{m-1}| \geqslant \max_{1 \leqslant j \leqslant m-1} |x_j|\,.
$$

With the latter observation one estimates, using (5.16),

$$
\begin{aligned}
&\int_{\complement B(|x|/2)} r_\mu(x_1) r_\mu(x_1 - x_2) \cdots r_\mu(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} \\
&\quad = \int_{\complement B(|x|/2)} e^{-\frac{\sqrt{\mu}}{2}\left(|x_1| + \sum_{j=1}^{m-2}|x_{j+1}-x_j| + |x_{m-1}|\right)} r_{\frac{\mu}{4}}(x_1) r_{\frac{\mu}{4}}(x_1 - x_2) \cdots r_{\frac{\mu}{4}}(x_{m-1}) \\
&\qquad \times d^n x_1 \cdots d^n x_{m-1} \\
&\quad \leqslant (2^{\widehat{n}-1})^m \int_{\complement B(|x|/2)} e^{-\frac{\sqrt{\mu}}{2}\left(\max_{j=1}^{m-1}|x_j|\right)} r_{\frac{\mu}{4}}(x_1) r_{\frac{\mu}{4}}(x_1 - x_2) \cdots r_{\frac{\mu}{4}}(x_{m-1}) \\
&\qquad \times d^n x_1 \cdots d^n x_{m-1} \\
&\quad \leqslant (2^{\widehat{n}-1})^m e^{-\frac{\sqrt{\mu}}{4}|x|} \int_{\complement B(|x|/2)} r_{\frac{\mu}{4}}(x_1) r_{\frac{\mu}{4}}(x_1 - x_2) \cdots r_{\frac{\mu}{4}}(x_{m-1}) \\
&\qquad \times d^n x_1 \cdots d^n x_{m-1} \\
&\quad \leqslant (2^{\widehat{n}-1})^m e^{-\frac{\sqrt{\mu}}{4}|x|} \int_{(\mathbb{R}^n)^{m-1}} r_{\frac{\mu}{4}}(x_1) r_{\frac{\mu}{4}}(x_1 - x_2) \cdots r_{\frac{\mu}{4}}(x_{m-1}) \\
&\qquad \times d^n x_1 \cdots d^n x_{m-1}.
\end{aligned}
$$

The latter expression decays faster than any power of $(1+|x|)^{-1}$. In fact, given Lemma 5.10, for $\sqrt{\mu} > 2\sum_{j=1}^{m+1} \alpha_j$, one obtains $e^{-\frac{\sqrt{\mu}}{4}|x|}[1+(|x|/2)]^{\sum_{j=1}^{m+1}\alpha_j} \leqslant 1$. Hence, for some $\kappa' > 0$,

$$
|t(x,x)| \leqslant (2^{\widehat{n}-1})^m \kappa' [1+(|x|/2)]^{-\sum_{j=1}^m \alpha_j}, \quad x \in \mathbb{R}^n.
$$

For the more precise estimate, one observes that

$$
\int_{(\mathbb{R}^n)^{m-1}} r_{\frac{\mu}{4}}(x_1) r_{\frac{\mu}{4}}(x_1 - x_2) \cdots r_{\frac{\mu}{4}}(x_{m-1})\, d^n x_1 \cdots d^n x_{m-1} = R_{\frac{\mu}{4}}^m \delta_{\{0\}}(0)
$$

and then applies Proposition 5.8 to estimate the latter expression. $\square$

*Remark* 5.15. A further inspection of Lemma 5.14 reveals that if $\sqrt{\mathrm{Re}(\mu)} > 2\ell$ for some $\ell \leqslant \sum_{j=1}^{m+1} \alpha_j$, one obtains the estimates

$$|t(x,x)| \leqslant \left[(2c_{\arg\mu})^{\widehat{n}-1}\right]^m \kappa_1 \cdots \kappa_m \left(\frac{4}{\mathrm{Re}(\mu)}\right)^m \left(\sqrt{\frac{\mathrm{Re}(\mu)}{4}}\right)^n c[1 + (|x|/2)]^{-\ell}, \quad x \in \mathbb{R}^n,$$

$$|t_k(x,x)| \leqslant c_{\arg\mu} \left[(c_{\arg\mu})^{\widehat{n}-1}\right]^m 2^{\widehat{n}\,m} \kappa_1 \cdots \kappa_{m+1} \times \\ \times \left(\frac{4}{\mathrm{Re}(\mu)}\right)^{m+1} \left(\sqrt{\frac{\mathrm{Re}(\mu)}{4}}\right)^{n+1} c'[1 + (|x|/2)]^{-\ell}, \quad x \in \mathbb{R}^n,$$

with $c$ and $c'$ given by (5.9) and (5.10), respectively.

To illustrate the importance of this result, envisage a product as in (5.22) with $m$ factors, all of them decaying like $|x|^{-1}$ as $|x| \to \infty$. Later on, we shall see that in certain integrals it suffices to estimate the diagonal decaying like $|x|^{-n}$ as $|x| \to \infty$. Thus, if $m$ is fairly large compared to $n$, and hence Lemma 5.14 yields a decay like $|x|^{-m}$, we have to choose the real-part of $\mu$ rather large as the explicit constant is only valid for $\sqrt{\mathrm{Re}(\mu)} > m$. But, if we are only interested in an estimate of the type $|x|^{-n}$, we may choose $\sqrt{\mathrm{Re}(\mu)}$ *a apriori* just larger than $n$. $\diamond$

A readily applicable version of Lemma 5.14 reads as follows.

**Lemma 5.16.** *Let $n = 2\widehat{n} + 1 \in \mathbb{N}_{\geqslant 3}$ odd, $m \geqslant \widehat{n} + 1$, assume that $\Psi_1, \ldots, \Psi_m \in C_b^\infty(\mathbb{R}^n)$, and suppose that $\mu \in \mathbb{C}_{\mathrm{Re}>0}$. Let $R_\mu$ and $Q$ be given by* (4.6) *and* (4.1)*, respectively. Assume that there exists $\alpha_j, \kappa_j \in \mathbb{R}_{\geqslant 0}$, $j \in \{1, \ldots, m\}$, such that*

$$|\Psi_j(x)| \leqslant \kappa_j (1 + |x|)^{-\alpha_j}, \quad x \in \mathbb{R}^n,\ j \in \{1, \ldots, m\}.$$

*Let $\ell \in \{2, \ldots, m\}$ and assume that there exists $\varepsilon \geqslant 0$ such that*

$$|(Q\Psi_\ell)(x)| + \left|\left(Q^2\Psi_\ell\right)(x)\right| \leqslant \kappa_\ell (1 + |x|)^{-\alpha_\ell - \varepsilon}, \quad x \in \mathbb{R}^n.$$

*If $t$ denotes the integral kernel of*

$$T := \prod_{j=1}^{\ell-2} (\Psi_j R_\mu)\, \Psi_{\ell-1} [R_\mu, \Psi_\ell]\, R_\mu \prod_{j=\ell+1}^{m} \Psi_j R_\mu$$

(*cf.* Remark 2.1 *and* (2.2)), *then $t$ is continuous on the diagonal and there exists $\kappa' > 0$ such that*

$$|t(x,x)| \leqslant \kappa' [1 + (|x|/2)]^{-\varepsilon - \sum_{j=1}^m \alpha_j}, \quad x \in \mathbb{R}^n.$$

*If $\sqrt{\mathrm{Re}(\mu)} > 2\sum_{j=1}^m \alpha_j + 2\varepsilon$, then a possible choice for $\kappa'$ is*

$$\kappa' = \kappa_1 \cdots \kappa_m \left(\frac{4}{\mathrm{Re}(\mu)}\right)^m \left(\sqrt{\frac{\mathrm{Re}(\mu)}{4}}\right)^n d, \tag{5.24}$$

*where $d := \left[(2c_{\arg\mu})^{\widehat{n}-1}\right]^m c + 2c_{\arg\mu}^{\widehat{n}} \left[(c_{\arg\mu})^{\widehat{n}-1}\right]^{m-1} 2^{\widehat{n}\,m} c'$, with $c$ and $c'$ given by* (5.9) *and* (5.10)*, respectively.*

*Proof.* One recalls from Lemma 4.4 (see also Remark 2.1) that

$$[R_\mu, \Psi_\ell] = R_\mu \left(Q^2\Psi_\ell\right) R_\mu + 2R_\mu (Q\Psi_\ell)\, Q R_\mu.$$

Let $t_1$ be the associated integral kernel of

$$\Psi_1 R_\mu \cdots \Psi_{\ell-1} R_\mu \left(Q^2 \Psi_\ell\right) R_\mu R_\mu \Psi_{\ell+1} R_\mu \cdots \Psi_m R_\mu$$

and $t_2$ the one of

$$\Psi_1 R_\mu \cdots \Psi_{\ell-1} R_\mu \left(Q \Psi_\ell\right) Q R_\mu R_\mu \Psi_{\ell+1} R_\mu \cdots \Psi_m R_\mu.$$

By hypothesis and by Lemma 5.14, for some constant $\kappa' > 0$,

$$|t_1(x,x)| + |t_2(x,x)| \leqslant \kappa'[1 + (|x|/2)]^{-\varepsilon - \sum_{j=1}^m \alpha_j}, \quad x \in \mathbb{R}^n. \tag{5.25}$$

The quantitative version of this assertion (i.e., the fact that $\kappa'$ given by (5.24) is a possible choice in the estimate (5.25)), also follows from Lemma 5.14. $\square$

Finally, we state one more variant of Lemma 5.14.

**Lemma 5.17.** *Let $n = 2\widehat{n} + 1 \in \mathbb{N}_{\geqslant 3}$ odd, $m \geqslant \widehat{n} + 1$, assume that $\Psi_1, \dots, \Psi_m \in C_b^\infty(\mathbb{R}^n)$, and $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) > 0$. Let $R_\mu$ be given by* (4.6), *and $Q$ by* (4.1). *Let $\varepsilon, \alpha_j, \kappa_j \in \mathbb{R}_{\geqslant 0}$, and assume that for all $j \in \{1, \dots, m\}$ and $\ell \in \{2, \dots, m\}$,*

$$|\Psi_j(x)| \leqslant \kappa_j (1 + |x|)^{-\alpha_j}, \ |(Q\Psi_\ell)(x)| + \left|\left(Q^2 \Psi_\ell\right)(x)\right| \leqslant \kappa_\ell [1 + (|x|/2)]^{-\alpha_\ell - \varepsilon}, \quad x \in \mathbb{R}^n.$$

*Then for $\ell \in \{1, \dots, m\}$, the associated integral kernels $h_\ell$ and $\widetilde{h}_\ell$ of*

$$\bigg(\prod_{j=1}^{\ell} \Psi_j R_\mu\bigg)\bigg(\prod_{j=\ell+1}^{m} \Psi_j\bigg) R_\mu^{m-\ell} \ \text{ and } \ \bigg(\prod_{j=1}^{\ell-1} \Psi_j R_\mu\bigg)\bigg(\prod_{j=\ell}^{m} \Psi_j\bigg) R_\mu^{m-\ell+1},$$

*respectively* (*cf. Remark* 2.1), *satisfy for some $\kappa' > 0$,*

$$\left|h_\ell(x,x) - \widetilde{h}_\ell(x,x)\right| \leqslant \kappa'[1 + (|x|/2)]^{-\varepsilon - \sum_{j=1}^m \alpha_j}, \quad x \in \mathbb{R}^n.$$

*In addition, if $\sqrt{\mathrm{Re}(\mu)} > 2\sum_{j=1}^m \alpha_j + 2\varepsilon$, then a possible choice for $\kappa'$ is given by* (5.24).

*Proof.* We will exploit Lemma 5.16 and note that $h_\ell - \widetilde{h}_\ell$ is the associated integral kernel of the operator

$$\begin{aligned}
&\bigg(\prod_{j=1}^{\ell} \Psi_j R_\mu\bigg)\bigg(\prod_{j=\ell+1}^{m} \Psi_j\bigg) R_\mu^{m-\ell} - \bigg(\prod_{j=1}^{\ell-1} \Psi_j R_\mu\bigg)\bigg(\prod_{j=\ell}^{m} \Psi_j\bigg) R_\mu^{m-\ell+1} \\
&\quad = \bigg(\bigg(\prod_{j=1}^{\ell-1} \Psi_j R_\mu\bigg)\Psi_\ell\bigg)\bigg(R_\mu \bigg(\prod_{j=\ell+1}^{m} \Psi_j\bigg) - \bigg(\prod_{j=\ell+1}^{m} \Psi_j\bigg) R_\mu\bigg) R_\mu^{m-\ell} \\
&\quad = \bigg(\bigg(\prod_{j=1}^{\ell-1} \Psi_j R_\mu\bigg)\Psi_\ell\bigg)\bigg(\bigg[R_\mu, \bigg(\prod_{j=\ell+1}^{m} \Psi_j\bigg)\bigg]\bigg) R_\mu^{m-\ell}.
\end{aligned}$$

By hypothesis,

$$\left|\bigg(\prod_{j=\ell+1}^{m} \Psi_j\bigg)(x)\right| \leqslant \kappa_{\ell+1} \cdots \kappa_m (1 + |x|)^{-\sum_{j=\ell+1}^m \alpha_j}, \quad x \in \mathbb{R}^n.$$

Moreover, by the product rule one concludes that

$$\left|\bigg(Q \prod_{j=\ell+1}^{m} \Psi_j\bigg)(x)\right| \leqslant \kappa_{\ell+1} \cdots \kappa_m (1 + |x|)^{-\varepsilon - \sum_{j=\ell+1}^m \alpha_j}, \quad x \in \mathbb{R}^n,$$

and thus also that

$$\left|\left(Q^2 \prod_{j=\ell+1}^{m} \Psi_j\right)(x)\right| \leqslant \kappa_{\ell+1} \cdots \kappa_m (1+|x|)^{-\varepsilon-\sum_{j=\ell+1}^{m} \alpha_j}, \quad x \in \mathbb{R}^n.$$

Hence, the assertion indeed follows from Lemma 5.16. □

*Remark* 5.18. Iterated application Lemma 5.17 shows that under the same assumptions, the integral kernels $h_m$ and $\widetilde{h}_1$ of

$$\left(\prod_{j=1}^{m} \Psi_j R_\mu\right) \text{ and } \left(\prod_{j=1}^{m} \Psi_j\right) R_\mu^m,$$

respectively, satisfy for some $\kappa' > 0$,

$$\left|h_m(x,x) - \widetilde{h}_1(x,x)\right| \leqslant \kappa' [1 + (|x|/2)]^{-\varepsilon-\sum_{j=1}^{m} \alpha_j}, \quad x \in \mathbb{R}^n.$$

◇

## 6. Dirac-Type Operators

In this section, we discuss the operator $L$ with a bounded smooth potential, studied by Callias [22] in $L^2(\mathbb{R}^n)^p$ for a suitable $p \in \mathbb{N}$. We compute its domain, its adjoint and give conditions for the Fredholm property of this operator.

Let

$$H_j^1(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) \,\middle|\, \partial_j f \in L^2(\mathbb{R}^n) \right\}, \quad j \in \{1, \dots, n\},$$

where $\partial_j f$ denotes the distributional partial derivative of $f \in L^2(\mathbb{R}^n)$ with respect to the $j$th variable. One notes that (see also (5.1))

$$H^1(\mathbb{R}^n) = \bigcap_{j \in \{1,\dots,n\}} H_j^1(\mathbb{R}^n).$$

*Remark* 6.1. In the following, we make use of the so-called *Euclidean Dirac algebra*, see Appendix A and Definition A.3 for the construction and some basic properties. For dimension $n \in \mathbb{N}$ we denote the elements of this algebra by $\gamma_{j,n}$, $j \in \{1, \dots, n\}$. One recalls that for $n = 2\widehat{n}$ or $n = 2\widehat{n} + 1$ for some $\widehat{n} \in \mathbb{N}$ one has

$$\gamma_{j,n}^* = \gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}} \times 2^{\widehat{n}}}, \quad \gamma_{j,n}\gamma_{k,n} + \gamma_{k,n}\gamma_{j,n} = 2\delta_{jk} I_{2^{\widehat{n}}}, \quad j,k \in \{1, \dots, n\}. \tag{6.1}$$

$\diamond$

We are now in the position to properly define the operator $L$ (and the underlying supersymmetric Dirac-type operator $H$) to be studied in the rest of this manuscript.

**Definition 6.2.** *Let $d \in \mathbb{N}$ and suppose that $\Phi\colon \mathbb{R}^n \to \mathbb{C}^{d \times d}$ is a bounded measurable function assuming values in the space of $d \times d$ self-adjoint matrices. We recall our convention $H^1(\mathbb{R}^n)^{2^{\widehat{n}}d} = H^1(\mathbb{R}^n)^{2^{\widehat{n}}} \otimes \mathbb{C}^d$. With this in mind, we introduce the* (*closed*) *operator $L$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ via*

$$L\colon \begin{cases} H^1(\mathbb{R}^n)^{2^{\widehat{n}}d} \subseteq L^2(\mathbb{R}^n)^{2^{\widehat{n}}d} \to L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}, \\ \psi \otimes \phi \mapsto \left(\sum_{j=1}^n \gamma_{j,n}\partial_j \psi\right) \otimes \phi + \left(x \mapsto \psi(x) \otimes \Phi(x)\phi\right). \end{cases} \tag{6.2}$$

*Henceforth, recalling* (4.1), *we shall abreviate*

$$\mathcal{Q} := Q \otimes I_d = \left(\sum_{j=1}^n \gamma_{j,n}\partial_j\right) I_d, \tag{6.3}$$

*and, with a slight abuse of notation, employ the symbol $\Phi$ also in the context of the operation*

$$\Phi\colon \psi \otimes \phi \mapsto \left(x \mapsto \psi(x) \otimes \Phi(x)\phi\right), \tag{6.4}$$

*see also our notational conventions to suppress tensor products whenever possible, collected in Section* 2 *and in Remark* 2.1. *Thus,*

$$L = \mathcal{Q} + \Phi. \tag{6.5}$$

*Finally, the underlying* (*self-adjoint*) *supersymmetric Dirac-type operator $H$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d} \oplus L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ is of the form*

$$H = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}. \tag{6.6}$$

A detailed treatment of supersymmetric Dirac-type operators can be found in [95, Ch. 5].

For clarity, we kept the tensor product notation in (6.2)–(6.4), but from now on we will typically dispense with tensor products to simplify notation.

In this section, we shall prove the following theorem:

**Theorem 6.3** ([22, Corollary on p. 217])**.** *Consider the operator $L$ given by* (6.2)*. Assume, in addition, that $\Phi \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$, $\Phi(x) = \Phi(x)^*$, $x \in \mathbb{R}^n$, and assume there exists $c > 0$ such that $|\Phi(x)| \geqslant cI_d$, $x \in \mathbb{R}^n$, as well as $(Q\Phi)(x) \to 0$ as $|x| \to \infty$. Then the operator*

$$L = \mathcal{Q} + \Phi, \quad \operatorname{dom}(L) = \operatorname{dom}(\mathcal{Q}) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}, \tag{6.7}$$

*is closed and Fredholm in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$. Consequently, the supersymmetric Dirac-type operator*

$$H = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}, \quad \operatorname{dom}(H) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}d} \oplus H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}, \tag{6.8}$$

*is self-adjoint and Fredholm in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d} \oplus L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$.*

In order to deduce Theorem 6.3, we need some preparations. The first result is concerned with the operator $Q$ in $L^2\left(\mathbb{R}^n\right)^{2^{\widehat{n}}}$ given by (4.1). Moreover, we will show that $Q$ is skew-self-adjoint and thus verify the estimate asserted in (4.4).

**Theorem 6.4.** *Let $n \in \mathbb{N}_{\geqslant 2}$ and hence $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}$, $j \in \{1,\dots,n\}$, with $n = 2\widehat{n}$ or $n = 2\widehat{n}+1$, see Remark* 6.1*. Denote*

$$\partial_j \colon H_j^1(\mathbb{R}^n)^{2^{\widehat{n}}} \subseteq L^2(\mathbb{R}^n)^{2^{\widehat{n}}} \to L^2(\mathbb{R}^n)^{2^{\widehat{n}}}, \quad f \mapsto \partial_j f,\ j \in \{1,\dots,n\}.$$

*Then the following assertions* $(i)$*–*$(iii)$ *hold:*

$(i)$ *$\partial_j$ is a skew-self-adjoint operator, $j \in \{1,\dots,n\}$.*

$(ii)$ *$\gamma_{j,n}\partial_j = \partial_j\gamma_{j,n}$ is skew-self-adjoint, $j \in \{1,\dots,n\}$.*

$(iii)$ *$Q = \sum_{j=1}^n \gamma_{j,n}\partial_j$, $\operatorname{dom}(Q) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}}$ is skew-self-adjoint* (*and thus closed*) *in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$.*

*Proof.* By Fourier transform (see (5.2)), the operator $\partial_j$ is unitarily equivalent to the operator given by multiplication by the function $x \mapsto ix_j$. The latter is a multiplication operator taking values on the imaginary axis; thus, it is skew-self-adjoint. Hence, so is $\partial_j$, proving assertion $(i)$.

Let $j \in \{1,\dots,n\}$. Assertion $(ii)$ follows from observing that $\gamma_{j,n}$ defines an isomorphism from $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ into itself. Indeed, this follows from the fact that $\gamma_{j,n}^2 = I_{2^{\widehat{n}}}$. Moreover, since $\gamma_{j,n}$ is a constant coefficient matrix it leaves the space $C_0^\infty(\mathbb{R}^n)^{2^{\widehat{n}}}$ invariant. The equality $\gamma_{j,n}\partial_j\phi = \partial_j\gamma_{j,n}\phi$ is clear for $\phi \in C_0^\infty(\mathbb{R}^n)^{2^{\widehat{n}}}$. Hence, $\partial_j\gamma_{j,n} \supseteq \gamma_{j,n}\partial_j$ and it remains to show that $C_0^\infty(\mathbb{R}^n)^{2^{\widehat{n}}}$ is a core for $\partial_j\gamma_{j,n}$. Let $\psi \in \operatorname{dom}(\partial_j\gamma_{j,n})$. Then there exists a sequence $\{\psi_k\}_{k\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^n)^{2^{\widehat{n}}}$ such that $\psi_k \to \gamma_{j,n}\psi$ as $k \to \infty$ in $D_{\partial_j}$, the domain of $\partial_j$ endowed with the graph norm. Defining $\phi_k := \gamma_{j,n}^{-1}\psi_k = \gamma_{j,n}\psi_k \in C_0^\infty(\mathbb{R}^n)^{2^{\widehat{n}}}$, $k \in \mathbb{N}$, one sees that $\phi_k \to \psi$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ and $\partial_j\gamma_{j,n}\phi_k = \partial_j\psi_k \to \partial_j\gamma_{j,n}\psi$ in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ as $k \to \infty$. Thus, $(\gamma_{j,n}\partial_j)^* = -\partial_j\gamma_{j,n}^* = -\partial_j\gamma_{j,n} = -\gamma_{j,n}\partial_j$.

Assertion $(iii)$ is a bit more involved. We shall prove it in the next two steps. $\square$

We recall the following well-known fact in the theory of normal operators.

**Theorem 6.5** (see, e.g., [52, p. 347])**.** *Let $\mathcal{H}$ be a complex separable Hilbert space and let $A$ and $B$ be self-adjoint, resolvent commuting operators acting on $\mathcal{H}$. Then*

$$A + iB$$

*is closed, densely defined, and*

$$(A + iB)^* = A - iB.$$

At this point we are ready to conclude the proof of Theorem 6.4:

**Lemma 6.6.** *Let $\mathcal{H}$ be a complex, separable Hilbert space, $A_1, \dots, A_n$ be resolvent commuting skew-self-adjoint operators in $\mathcal{H}$. Let $\{\gamma_k\}_{k\in\{1,\dots,n\}}$ be a family of bounded linear self-adjoint operators in $\mathcal{H}$, all commuting with $A_j$, $j \in \{1, \dots, n\}$, in the sense that $\gamma_k A_j = A_j \gamma_k$, $j, k \in \{1, \dots, n\}$. Assume that the following equation holds,*

$$\gamma_k \gamma_{k'} + \gamma_{k'} \gamma_k = 2\delta_{kk'}, \quad k, k' \in \{1, \dots, n\}.$$

*Then $\sum_{k=1}^n \gamma_k A_k$ is closed on its natural domain $\bigcap_{k=1}^n \operatorname{dom}(A_k)$, and*

$$\left(\sum_{k=1}^n \gamma_k A_k\right)^* = -\sum_{k=1}^n \gamma_k A_k. \tag{6.9}$$

*Proof.* We prove (6.9) by induction on $n$. The case $n = 1$ follows from $(\gamma_1 A_1)^* = A_1^* \gamma_1^* = -A_1 \gamma_1 = -\gamma_1 A_1$.

Next, assume the assertion holds for $n \in \mathbb{N}$ and consider the sum

$$A := \sum_{k=1}^{n+1} \gamma_k A_k = \gamma_1 A_1 + \sum_{k=2}^{n+1} \gamma_k A_k,$$

with its natural domain $\bigcap_{k=1}^n \operatorname{dom}(A_k)$. Since $\gamma_1^2 = I_{\mathcal{H}}$, $\gamma_1$ defines an isomorphism from $\mathcal{H}$ into itself. Hence, $A$ is closed if and only if $\gamma_1 A$ is closed. One notes,

$$\left(\gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k\right)^* = -\sum_{k=2}^{n+1} \gamma_k A_k \gamma_1 = -\sum_{k=2}^{n+1} \gamma_k \gamma_1 A_k = \sum_{k=2}^{n+1} \gamma_1 \gamma_k A_k = \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k, \tag{6.10}$$

in addition, $\gamma_1 \gamma_1 A_1 = A_1$ is skew-self-adjoint. With the help of Theorem 6.5 it remains to check whether the resolvents of $A_1$ and $\gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k$ commute. One observes that for $z \in \varrho(A_1)$,

$$\begin{aligned}
(z - A_1)^{-1} \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k &= \gamma_1 (z - A_1)^{-1} \sum_{k=2}^{n+1} \gamma_k A_k = \gamma_1 \sum_{k=2}^{n+1} \gamma_k (z - A_1)^{-1} A_k \\
&\subseteq \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k (z - A_1)^{-1}.
\end{aligned} \tag{6.11}$$

Adding $-z' (z - A_1)^{-1}$ for some $z' \in \varrho\left(\gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k\right)$ to both sides of inclusion (6.11), one obtains

$$-z' (z - A_1)^{-1} + (z - A_1)^{-1} \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k$$

$$\subseteq -z' \left(z - A_1\right)^{-1} + \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k \left(z - A_1\right)^{-1}.$$

Thus,

$$\left(z - A_1\right)^{-1} \left(z' - \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k\right) \subseteq \left(z' - \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k\right) \left(z - A_1\right)^{-1},$$

implying

$$\left(z' - \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k\right)^{-1} \left(z - A_1\right)^{-1} \subseteq \left(z - A_1\right)^{-1} \left(z' - \gamma_1 \sum_{k=2}^{n+1} \gamma_k A_k\right)^{-1},$$

proving assertion (6.9) since the domain of the operator on the left-hand side is all of $\mathcal{H}$. $\square$

For proving the Fredholm property of $L = \mathcal{Q} + \Phi$, we will employ stability of the Fredholm property under relatively compact perturbations, or, in other words, that the essential spectrum is invariant under additive relatively compact perturbations. Thus, we need a compactness criterion and hence we recall the following compactness result for multiplication operators, a consequence of the Rellich–Kondrachov theorem, see [2, Theorem 6.3] (cf. and (5.1) for the definition of $H^1(\mathbb{R}^n)$).

**Theorem 6.7.** *Let $n \in \mathbb{N}$ and $\phi \in L^\infty(\mathbb{R}^n)$ such that for all $\varepsilon > 0$ there exists $\Lambda > 0$ such that for all $x \in \mathbb{R}^n \backslash B(0, \Lambda)$, $|\phi(x)| \leqslant \varepsilon$. Then*

$$\phi\colon \begin{cases} H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\ f \mapsto \phi(\cdot) f(\cdot), \end{cases} \qquad \textit{is compact.}$$

*Proof.* As $H^1(\mathbb{R}^n)$ is a Hilbert space, it suffices to prove that weakly convergent sequences are mapped to norm-convergent sequences: Suppose that $\{f_k\}_{k\in\mathbb{N}}$ weakly converges to some $f$ in $H^1(\mathbb{R}^n)$ and denote $M \coloneqq \sup_{k\in\mathbb{N}} \|f_k\|_{H^1(\mathbb{R}^n)}$, which is finite by the uniform boundedness principle. In particular, $\{f_k\}_{k\in\mathbb{N}}$ converges weakly in $H^1(B(0,\Lambda))$ for every $\Lambda > 0$. Hence, by the Rellich–Kondrachov theorem, for all $\Lambda > 0$ the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges in $L^2(B(0,\Lambda))$. Next, let $\varepsilon > 0$. As $f \in L^2(\mathbb{R}^n)$, there exists $\Lambda_0 > 0$ such that $\|f\chi_{B(0,\Lambda_0)} - f\|_{L^2} \leqslant \varepsilon$, where we denoted by $\chi_{B(0,\Lambda_0)}$ the cut-off function being 1 on the ball $B(0,\Lambda_0)$ and 0 elswhere. One can find $\Lambda \geqslant \Lambda_0$ such that $|\phi(x)| \leqslant \varepsilon$ for $|x| \geqslant \Lambda$, and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, one has $\|f_k\chi_{B(0,\Lambda)} - f\chi_{B(0,\Lambda)}\|_{L^2} \leqslant \varepsilon$. Thus, for $k \geqslant k_0$ one arrives at

$$\begin{aligned} \|\phi f_k - \phi f\|^2 &= \left\|\phi f_k \chi_{B(0,\Lambda)} - \phi f \chi_{B(0,\Lambda)}\right\|_{L^2}^2 + \left\|\phi f_k \chi_{\mathbb{R}^n\backslash B(0,\Lambda)} - \phi f \chi_{\mathbb{R}^n\backslash B(0,\Lambda)}\right\|_{L^2}^2 \\ &\leqslant \|\phi\|_{L^\infty}^2 \varepsilon^2 + \varepsilon^2 \left(2M\right)^2. \end{aligned} \tag{6.12}$$

$\square$

*Remark* 6.8. The latter theorem has the following easy but important corollary: In the situation of Theorem 6.7, let $\mathcal{H}$ be a Hilbert space continuously embedded into $H^1(\mathbb{R}^n)$, for instance, $\mathcal{H} = H^2(\mathbb{R}^n)$ (cf. (5.1)), then the operator $\phi_{\mathcal{H}\to L^2}$ of multiplying by $\phi$ considered from $\mathcal{H}$ to $L^2(\mathbb{R}^n)$ is compact. Denoting by $\iota\colon \mathcal{H} \to H^1(\mathbb{R}^n)$ the continuous embedding, which exists by hypothesis, one observes that

$$\phi_{\mathcal{H}\to L^2(\mathbb{R}^n)} = \phi_{H^1(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \circ \iota,$$

with $\phi_{H^1(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$ being the operator discussed in Theorem 6.7. Hence, the operator $\phi_{\mathcal{H}\to L^2(\mathbb{R}^n)}$ is compact as a composition of a continuous and a compact operator. $\diamond$

The proof of Theorem 6.3 will rest on the observation that $L$ is Fredholm if and only if the essential spectra of $L^*L$ and $LL^*$ have strictly positive lower bounds. Thus, we formulate two propositions describing the opertors $L^*L$ and $LL^*$ in bit more detail:

**Proposition 6.9.** *The operator $L$ given by* (6.2) *is closed and densely defined in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ and*

$$L^* = -\mathcal{Q} + \Phi, \quad \operatorname{dom}(L^*) = \operatorname{dom}(L) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}. \tag{6.13}$$

*Proof.* Since the operator of multiplication with the function $\Phi$ is bounded and self-adjoint, the assertion is immediate from Theorem 6.4. $\square$

**Proposition 6.10.** *Assume that $\Phi \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ is pointwise self-adjoint, that is, $\Phi(\cdot) = \Phi(\cdot)^*$. For $L = \mathcal{Q} + \Phi$ given by* (6.2)*, one then has* (*cf.* (6.4))*,*

$$L^*L = -\Delta I_{2^{\widehat{n}}d} - C + \Phi^2 \ \text{ and } \ LL^* = -\Delta I_{2^{\widehat{n}}d} + C + \Phi^2, \tag{6.14}$$

*where*

$$C = \sum_{j=1}^{n} \gamma_{j,n}(\partial_j \Phi) = (\mathcal{Q}\Phi), \tag{6.15}$$

*see also Remark* 2.1*. Moreover,*

$$\operatorname{dom}(L^*L) = \operatorname{dom}(LL^*) = H^2(\mathbb{R}^n)^{2^{\widehat{n}}d} \tag{6.16}$$

(*see* (5.1) *for a definition of the latter*)*.*

*Proof.* At first one observes that if $\psi \in H^k(\mathbb{R}^n)^{2^{\widehat{n}}d}$ and $L\psi \in H^k(\mathbb{R}^n)^{2^{\widehat{n}}d}$, then $\psi \in H^{k+1}(\mathbb{R}^n)^{2^{\widehat{n}}d}$. Indeed, from $L\psi = \mathcal{Q}\psi + \Phi\psi$, one infers $\psi + \mathcal{Q}\psi = \psi + L\psi + \Phi\psi \in H^k(\mathbb{R}^n)^{2^{\widehat{n}}d}$ by the differentiablity of $\Phi$. By Theorem 6.4, the operator $\mathcal{Q}$ is skew-self-adjoint and therefore $-1 \in \varrho(\mathcal{Q})$. Hence, $\psi = (\mathcal{Q}+I)^{-1}(\mathcal{Q}+I)\psi \in H^{k+1}(\mathbb{R}^n)^{2^{\widehat{n}}d}$. Therefore, if $\psi \in \operatorname{dom}(L) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}$ with $L\psi \in \operatorname{dom}(L^*) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}$, then $\psi \in H^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$. On the other hand, if $\psi \in H^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$, then also $\psi \in \operatorname{dom}(L^*L)$. The same reasoning applies to $LL^*$.

Next, we compute $L^*L$. With Proposition 6.9 one obtains

$$L^*L = (-\mathcal{Q} + \Phi)(\mathcal{Q} + \Phi) = -\mathcal{Q}\mathcal{Q} + \Phi\mathcal{Q} - \mathcal{Q}\Phi + \Phi^2$$

and

$$LL^* = (\mathcal{Q} + \Phi)(-\mathcal{Q} + \Phi) = -\mathcal{Q}\mathcal{Q} - \Phi\mathcal{Q} + \mathcal{Q}\Phi + \Phi^2.$$

Recalling $-\mathcal{Q}\mathcal{Q} = -\Delta I_{2^{\widehat{n}}d}$ from (4.3), one concludes the proof with the observation $\Phi\mathcal{Q} - \mathcal{Q}\Phi = \Phi\mathcal{Q} - \Phi\mathcal{Q} + C = C$, applying the product rule. $\square$

We may now come to the proof of the Fredholm property of $L = \mathcal{Q} + \Phi$ with smooth potential $\Phi$ satisfying for some $c > 0$, $|\Phi(x)| \geqslant cI_d$, $x \in \mathbb{R}^n$, as well as satisfying $C(x) = (\mathcal{Q}\Phi)(x) \to 0$ as $|x| \to \infty$:

*Proof of Theorem* 6.3. By hypothesis, $(\Phi(x))^2 = |\Phi(x)|^2 \geqslant c^2 I_d$, $x \in \mathbb{R}^n$. From

$$-\Delta I_d + \Phi^2 \geqslant c^2 I_d,$$

one deduces that the spectrum of $-\Delta I_d + \Phi^2$ is contained in $[c^2, \infty)$. In particular, one concludes that the essential spectrum $\sigma_{\mathrm{ess}}(-\Delta I_d + \Phi^2)$ of $-\Delta I_d + \Phi^2$ is also contained in $[c^2, \infty)$. Since $x \mapsto C(x) = (Q\Phi)(x)$ satisfies the condition imposed on $\Phi$ in Theorem 6.7, one infers that $C$ is $-\Delta I_d + \Phi^2$-compact, since the domain of the latter (closed) operator coincides with $H^2(\mathbb{R}^n)^d$, which is continuously embedded into $H^1(\mathbb{R}^n)^d$. Recalling Proposition 6.10, that is,

$$L^*L = -\Delta I_{2^{\hat{n}}d} - C + \Phi^2,$$

one obtains $\sigma_{\mathrm{ess}}(L^*L) = \sigma_{\mathrm{ess}}(-\Delta I_d + \Phi^2) \subseteq [c^2, \infty)$, as the essential spectrum is invariant under additive relatively compact perturbations (see, e.g., [71, Theorem 5.35]). In particular, $0 \notin \sigma_{\mathrm{ess}}(L^*L)$ implying that $L^*L$ is Fredholm. By a similar argument applied to $LL^*$, one deduces the Fredholm property of $L$ (using that $\ker(L) = \ker(L^*L)$ and $\ker(L^*) = \ker(LL^*)$). ☐

In the following sections, we are interested in a particular subclass of potentials $\Phi$. In particular, we focus on potentials for which we may apply Theorem 3.4. A first main focus is set on potentials satisfying the properties stated in Definition 6.11, the so-called *admissible* potentials. The reader is referred to Section 10 and beyond for possible generalizations. It should be noted, however, that for more general potentials the derivations and arguments are more involved than for the ones mentioned in Definition 6.11. In fact, the main reason being assumption $(ii)$ on the invertibility of $\Phi$ everywhere. It is known (see the end of Section 10) that the operator $L = \mathcal{Q} + \Phi$ has index 0 for $\Phi$ satisfying Definition 6.11. Later on, we shall see that the study of potentials being invertible on complements of large balls around 0 can be reduced to the study of potentials being invertible everywhere except on a sufficiently small ball around 0. The arguments for the latter case, in turn, rest on the perturbation theory for the Helmholtz equation, see Section 11. Hence, the derivation for the index formula for potentials being invertible everywhere except on a sufficiently small ball can be regarded as a perturbed version of the arguments given for admissible potentials. Therefore, we chose to present the core arguments for the by far simpler case of admissible potentials first.

The precise notion of what we call *admissible potentials* reads as follows.

**Definition 6.11.** *Let* $\Phi\colon \mathbb{R}^n \to \mathbb{C}^{d\times d}$ *for some* $d, n \in \mathbb{N}$. *We call* $\Phi$ admissible*, if the following conditions* $(i)$*–*$(iii)$ *hold:*

$(i)$ (*smoothness*) $\Phi \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$.

$(ii)$ (*invertibility and self-adjointness*) *for all* $x \in \mathbb{R}^n$, $\Phi(x)^* = \Phi(x) = \Phi(x)^{-1}$.

$(iii)$ (*asymptotics of the derivatives*) *for all* $\alpha \in \mathbb{N}_0^n$, *there exists* $\kappa > 0$ *and* $\varepsilon > 1/2$ *such that*

$$\|(\partial^\alpha \Phi)(x)\| \leqslant \begin{cases} \kappa(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa(1+|x|)^{-1-\varepsilon}, & |\alpha| \geqslant 2, \end{cases} \quad x \in \mathbb{R}^n,$$

*where we employed multi-index notation and used the convention* $|\alpha| = \sum_{j=1}^n \alpha_j$.

*Remark* 6.12. If $\Psi \in C_b^\infty\big(\mathbb{R}^n \backslash B(0,1); \mathbb{C}^{d\times d}\big)$ is *homogeneous of order* 0, that is, for all $x \in \mathbb{R}^n \backslash \{0\}$, $\Psi(x) = \Psi(x/|x|)$, then $\Psi$ satisfies Definition 6.11 $(iii)$. Indeed, one

computes for $x = \{x_j\}_{j\in\{1,\dots,n\}} \in \mathbb{R}^n\backslash\{0\}$ and $j \in \{1,\dots,n\}$,

$$\partial_j \left(\frac{\cdot}{|\cdot|}\right)(x) = \frac{1}{|x|}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - \frac{x_j}{|x|^3}\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix},$$

and

$$\begin{aligned}
(\partial_j \Psi)(x) &= \partial_j \left(\Psi \circ \left(\frac{\cdot}{|\cdot|}\right)\right)(x) \\
&= \begin{pmatrix} (\partial_1 \Psi)(x/|x|) & \cdots & (\partial_j \Psi)(x/|x|) & \cdots & (\partial_n \Psi)(x/|x|) \end{pmatrix} \\
&\quad \times \left( \frac{1}{|x|}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - \frac{x_j}{|x|^3}\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}\right) \\
&= \frac{1}{|x|}\sum_{k=1}^n (\partial_k \Psi)\left(\frac{x}{|x|}\right)\left(\delta_{kj} - \frac{x_k x_j}{|x|^2}\right),
\end{aligned}$$

establishing the assertion. We note that Callias [22] assumes that the potential "approaches a homogeneous function of order 0 as $|x| \to \infty$" such that Definition 6.11 $(iii)$ is satisfied. $\diamond$

## 7. Derivation of the Trace Formula – The Trace Class Result

In this section, we shall prove the applicability of Theorem 3.4 for the operator

$$L = \mathcal{Q} + \Phi \tag{7.1}$$

in $L^2(\mathbb{R}^n)^{2^{\hat{n}}d}$ as introduced in (6.2) with

$$\mathcal{Q} = \sum_{j=1}^{n} \gamma_{j,n}\partial_j$$

given by (6.3) (or (4.1)) and an admissible potential $\Phi$, see Definition 6.11. More precisely, we seek to establish that the operator

$$\chi_\Lambda B_L(z) = z\chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big), \quad z \in \varrho(-LL^*) \cap \varrho(-L^*L), \tag{7.2}$$

belongs to the trace class $\mathcal{B}_1\big(L^2(\mathbb{R}^n)\big)$, where $\operatorname{tr}_{2^{\hat{n}}d}$ is given in (3.1) and $\chi_\Lambda$ is the multiplication operator of multiplying with the characteristic function of the ball centered at 0 with radius $\Lambda > 0$, that is,

$$\chi_\Lambda(x) := \begin{cases} 1, & x \in B(0,\Lambda), \\ 0, & x \in \mathbb{R}^n \backslash B(0,\Lambda). \end{cases} \tag{7.3}$$

Regarding Theorem 3.4 (with $T_\Lambda = \chi_\Lambda$ and $S_\Lambda^* = I_{L^2(\mathbb{R}^n)}$), we are then interested in computing the limit for $\Lambda \to \infty$ of $\operatorname{tr}_{L^2(\mathbb{R}^n)}(\chi_\Lambda B_L(z))$. This requires showing that $\chi_\Lambda B_L(z)$ is indeed trace class for all $\Lambda > 0$. The limit $z \to 0$ of $\lim_{\Lambda\to\infty} \operatorname{tr}_{L^2(\mathbb{R}^n)}(\chi_\Lambda B_L(z))$ (provided it exists in an appropriate way, see (3.5) in Theorem 3.4) then corresponds to the index of $L$. It turns out that to compute the limit of $z \to 0$ in the expression $\lim_{\Lambda\to\infty} \operatorname{tr}_{L^2(\mathbb{R}^n)}(\chi_\Lambda B_L(z))$ is rather straightforward (see also Theorem 10.1), once the respective formula is established[2]. The main theorem, which we shall prove in the next two sections, reads as follows.

**Theorem 7.1.** *Let $z \in \varrho(-LL^*) \cap \varrho(-L^*L)$ with* $\operatorname{Re}(z) > -1$ *and $n \in \mathbb{N}_{\geqslant 3}$ odd. Suppose that $\Phi$ is admissible* (*see Definition* 6.11)*. Then the operator $\chi_\Lambda B_L(z)$ with $B_L(z)$ and $\chi_\Lambda$ given by* (7.2) *and* (7.3)*, respectively, is trace class, the limit $f(z) := \lim_{\Lambda\to\infty} \operatorname{tr}(\chi_\Lambda B_L(z))$ exists and is given by*

$$\begin{aligned} f(z) = (1+z)^{-n/2}\left(\frac{i}{8\pi}\right)^{(n-1)/2} & \frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}} \\ & \times \int_{\Lambda S^{n-1}} \operatorname{tr}(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x))x_j\, d^{n-1}\sigma(x), \end{aligned} \tag{7.4}$$

*where $\varepsilon_{ji_1\dots i_{n-1}}$ denotes the $\varepsilon$-symbol as in Proposition* A.8.

In order to deduce the latter theorem, we shall have a deeper look into the inner structure of $B_L(z)$. A first step toward our goal is the following result.

[2]From now on, we shall *only* furnish the internal trace, introduced in Definition 3.1, of operators living on an orthogonal sum of a Hilbert space, with an additional subscript. The operator tr without subscript will always refer to the trace of a trace class operator acting in some fixed underlying Hilbert space. In particular, for $A \in \mathbb{C}^{d\times d}$, the expression $\operatorname{tr}(A)$ denotes the sum of the diagonal entries.

**Lemma 7.2.** *Let $L$ and $B_L(z)$ be given by* (7.1) *and* (7.2)*, respectively. Then for all $z \in \varrho(-LL^*) \cap \varrho(-L^*L)$,*

$$2B_L(z) = \mathrm{tr}_{2^{\widehat{n}}d}\left(\left[L, L^*\left(LL^*+z\right)^{-1}\right]\right) - \mathrm{tr}_{2^{\widehat{n}}d}\left(\left[L^*, L\left(L^*L+z\right)^{-1}\right]\right)$$

(*where $[\cdot,\cdot]$ represents the commutator symbol, cf.* (2.2)).

*Proof.* Let $z \in \varrho\left(-LL^*\right) \cap \varrho\left(-L^*L\right)$. One computes

$$\begin{aligned}
[L, L^*\left(LL^*+z\right)^{-1}] &= LL^*\left(LL^*+z\right)^{-1} - L^*\left(LL^*+z\right)^{-1}L \\
&= \left(LL^*+z\right)\left(LL^*+z\right)^{-1} - z\left(LL^*+z\right)^{-1} - \left(L^*L+z\right)^{-1}L^*L \\
&= 1 - z\left(LL^*+z\right)^{-1} - \left(L^*L+z\right)^{-1}\left(L^*L+z\right) + \left(L^*L+z\right)^{-1}z \\
&= 1 - z\left(LL^*+z\right)^{-1} - 1 + \left(L^*L+z\right)^{-1}z \\
&= z\left(L^*L+z\right)^{-1} - z\left(LL^*+z\right)^{-1},
\end{aligned}$$

and, interchanging the roles of $L$ and $L^*$, one concludes

$$[L^*, L\left(L^*L+z\right)^{-1}] = z\left(LL^*+z\right)^{-1} - z\left(L^*L+z\right)^{-1}. \qquad \square$$

The forthcoming Proposition 7.4 gives a more detailed description of the commutators describing $B_L(z)$ just derived in Lemma 7.2. First, we need a prerequisit of a more general nature.

**Lemma 7.3.** *Let $n \in \mathbb{N}$, $B \in \mathcal{B}\big(L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}, L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}\big)$ and let $\mathcal{Q}$ and $\gamma_{j,n}$, $j \in \{1,\dots,n\}$, as in* (6.3) *and in Remark* 6.1*, respectively. Then, on the common natural domain of the operator sums involved, one has*

$$\mathrm{tr}_{2^{\widehat{n}}d}([\mathcal{Q}, B]) = \sum_{j=1}^{n} \mathrm{tr}_{2^{\widehat{n}}d}([\partial_j, \gamma_{j,n}B]) = \sum_{j=1}^{n} \mathrm{tr}_{2^{\widehat{n}}d}([\partial_j, B\gamma_{j,n}]).$$

*Proof.* One computes with the help of Proposition 3.3 and the fact $\gamma_{j,n}\partial_j = \partial_j\gamma_{j,n}$,

$$\begin{aligned}
\mathrm{tr}_{2^{\widehat{n}}d}(\mathcal{Q}B - B\mathcal{Q}) &= \sum_{j=1}^{n} \mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\partial_j B - B\gamma_{j,n}\partial_j\right) \\
&= \sum_{j=1}^{n} \left[\mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\partial_j B\right) - \mathrm{tr}_{2^{\widehat{n}}d}\left(\left(B\gamma_{j,n}\right)\partial_j\right)\right] \\
&= \sum_{j=1}^{n} \left[\mathrm{tr}_{2^{\widehat{n}}d}\left(\partial_j\gamma_{j,n} B\right) - \mathrm{tr}_{2^{\widehat{n}}d}(\left(\gamma_{j,n}B\right)\partial_j)\right] \\
&= \sum_{j=1}^{n} \mathrm{tr}_{2^{\widehat{n}}d}([\partial_j, \gamma_{j,n}B]).
\end{aligned}$$

The second equality can be shown similarly. $\square$

The following proposition represents the core of the derivation of the index formula. Once it is proven that $\chi_\Lambda B_L(z)$ is trace class, with the trace being computed as the integral over the diagonal of the respective integral kernel, equation (7.5) will be the key for computing the trace. More precisely, the first summand is a sum of commutators of certain operators with partial derivatives. For the respective integral kernels, this will give us an expression as in Lemma 5.6 (see also (5.5)), which will enable us to use Gauss' divergence theorem, explaining the surface integral in

(7.4). Furthermore, the second summand in equation (7.5) as can be seen in equation (7.7) is basically a commutator of an integral operator and a multiplication operator. The integral kernels of this type of operators have been shown to vanish on the diagonal in Proposition 5.5, thus, (7.7) will give a vanishing contribution to the trace of $B_L(z)$.

**Proposition 7.4** ([22, Proposition 1, p. 219])**.** *Let $L$ be given by* (7.1) *and $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$. Then $B_L(z)$ given by* (7.2) *satisfies*

$$2B_L(z) = \sum_{j=1}^{n} \big[\partial_j, J_L^j(z)\big] + A_L(z), \tag{7.5}$$

*where*

$$J_L^j(z) = \operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right) + \operatorname{tr}_{2^{\hat{n}}d}\left(L^*\left(LL^*+z\right)^{-1}\gamma_{j,n}\right), \quad j \in \{1,\dots,n\}, \tag{7.6}$$

*and*

$$A_L(z) = \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\Phi, L^*(LL^*+z)^{-1}\big]\right) - \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\Phi, L(L^*L+z)^{-1}\big]\right), \tag{7.7}$$

*with $\gamma_{j,n}$ as in Remark* 6.1 *or Appendix* A.

*Proof.* One recalls that $L^* = -\mathcal{Q} + \Phi$ from Proposition 6.9. From Lemma 7.2, one infers that

$$\begin{aligned}
2B_L(z) &= \operatorname{tr}_{2^{\hat{n}}d}\left(\big[L, L^*(LL^*+z)^{-1}\big]\right) - \operatorname{tr}_{2^{\hat{n}}d}\left(\big[L^*, L(L^*L+z)^{-1}\big]\right)\\
&= \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\mathcal{Q}+\Phi, L^*(LL^*+z)^{-1}\big]\right) - \operatorname{tr}_{2^{\hat{n}}d}\left(\big[-\mathcal{Q}+\Phi, L(L^*L+z)^{-1}\big]\right)\\
&= \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\mathcal{Q}, L^*(LL^*+z)^{-1}\big]\right) + \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\mathcal{Q}, L(L^*L+z)^{-1}\big]\right)\\
&\quad + \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\Phi, L^*(LL^*+z)^{-1}\big]\right) - \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\Phi, L(L^*L+z)^{-1}\big]\right).
\end{aligned}$$

The equations

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\big[\mathcal{Q}, L^*(LL^*+z)^{-1}\big]\right) = \sum_{j=1}^{n} \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\partial_j, L^*(LL^*+z)^{-1}\gamma_{j,n}\big]\right),$$

and

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\big[\mathcal{Q}, L(L^*L+z)^{-1}\big]\right) = \sum_{j=1}^{n} \operatorname{tr}_{2^{\hat{n}}d}\left(\big[\partial_j, L(L^*L+z)^{-1}\gamma_{j,n}\big]\right)$$

follow from Lemma 7.3. □

Next, we show that (a modification in the sense of Theorem 3.4 of) $B_L(z)$ gives rise to trace class operators. Before doing so in Theorem 7.8, we need a different representation of $B_L(z)$ in terms of powers of the resolvent of the (free) Laplacian. One notes that for $z \in \mathbb{C}$, with $\operatorname{Re}(z) > \sup_{x\in\mathbb{R}^n} \max_j \|\partial_j \Phi(x)\| - 1$, one has $\|CR_{1+z}\| < 1$, with $C$ given by (6.15). Hence, by Proposition 6.10, equation (6.14), one obtains

$$\begin{aligned}
\left(L^*L+z\right)^{-1} &= \left(-\Delta I_{2^{\hat{n}}d} - C + (1+z)\right)^{-1}\\
&= \left(\left(-\Delta I_{2^{\hat{n}}d} + (1+z)\right)\left(I_{2^{\hat{n}}d} - R_{1+z}C\right)\right)^{-1}\\
&= \left(I_{2^{\hat{n}}d} - R_{1+z}C\right)^{-1} R_{1+z}\\
&= \sum_{k=0}^{\infty} \left(R_{1+z}C\right)^k R_{1+z},,
\end{aligned} \tag{7.8}$$

and, similarly,

$$(LL^*+z)^{-1} = (-\Delta I_{2^{\hat{n}}d} + C + (1+z))^{-1} = \sum_{k=0}^{\infty} (-R_{1+z}C)^k \, R_{1+z}. \tag{7.9}$$

Consequently, by analytic continuation, one obtains for $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$ with $\mathrm{Re}(z) > -1$,

$$(L^*L+z)^{-1} = \sum_{k=0}^{N} (R_{1+z}C)^k \, R_{1+z} + (R_{1+z}C)^{N+1} \, (L^*L+z)^{-1}, \tag{7.10}$$

and

$$(LL^*+z)^{-1} = \sum_{k=0}^{N} (-R_{1+z}C)^k \, R_{1+z} + (-R_{1+z}C)^{N+1} \, (LL^*+z)^{-1}, \tag{7.11}$$

for all $N \in \mathbb{N}$. Focussing on resolvent differences, one gets the following proposition:

**Proposition 7.5.** *Let $z \in \mathbb{C}_{\mathrm{Re}>-1}$. One recalls $L = \mathcal{Q} + \Phi$ as in* (7.1)*, $C = (\mathcal{Q}\Phi)$ from* (6.15)*, and $R_{1+z}$ in* (4.6)*.*
*$(i)$ If $\mathrm{Re}(z) > \sup_{x\in\mathbb{R}^n} \max_j \|(\partial_j\Phi)(x)\| - 1$, then $z \in \varrho\,(-L^*L) \cap (-LL^*)$ and*

$$(L^*L+z)^{-1} - (LL^*+z)^{-1} = 2\sum_{k=0}^{\infty} (R_{1+z}C)^{2k+1} \, R_{1+z} = 2\sum_{k=0}^{\infty} R_{1+z} \, (CR_{1+z})^{2k+1},$$

*as well as*

$$(L^*L+z)^{-1} + (LL^*+z)^{-1} = 2\sum_{k=0}^{\infty} (R_{1+z}C)^{2k} \, R_{1+z} = 2\sum_{k=0}^{\infty} R_{1+z} \, (CR_{1+z})^{2k}.$$

*$(ii)$ If $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$ and $\mathrm{Re}(z) > -1$, then for all $N \in \mathbb{N}$,*

$$\begin{aligned} &(L^*L+z)^{-1} - (LL^*+z)^{-1} \\ &\quad = 2\sum_{k=0}^{N} R_{1+z} \, (CR_{1+z})^{2k+1} + \big( (L^*L+z)^{-1} - (LL^*+z)^{-1} \big) \, (CR_{1+z})^{2N+2} \\ &\quad = 2\sum_{k=0}^{N} R_{1+z} \, (CR_{1+z})^{2k+1} + \big( (L^*L+z)^{-1} + (LL^*+z)^{-1} \big) \, (CR_{1+z})^{2N+3}, \end{aligned}$$

*and*

$$\begin{aligned} &(L^*L+z)^{-1} + (LL^*+z)^{-1} \\ &\quad = 2\sum_{k=0}^{N} R_{1+z} \, (CR_{1+z})^{2k} + \big( (L^*L+z)^{-1} + (LL^*+z)^{-1} \big) \, (CR_{1+z})^{2N+2}. \end{aligned}$$

*Proof.* $(i)$ This is a direct consequence of equations (7.8) and (7.9).
$(ii)$ For $z$ as in part $(i)$ one computes, similarly to (7.10) and (7.11), with the help of item $(i)$ for $N \in \mathbb{N}$,

$$\begin{aligned} &(L^*L+z)^{-1} - (LL^*+z)^{-1} \\ &\quad = 2\sum_{k=0}^{N} R_{1+z} \, (CR_{1+z})^{2k+1} + 2\sum_{k=N+1}^{\infty} R_{1+z} \, (CR_{1+z})^{2k+1} \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=0}^{N} R_{1+z}\left(CR_{1+z}\right)^{2k+1} + 2\sum_{k=0}^{\infty} R_{1+z}\left(CR_{1+z}\right)^{2k+2N+2+1} \\
&= 2\sum_{k=0}^{N} R_{1+z}\left(CR_{1+z}\right)^{2k+1} + 2\sum_{k=0}^{\infty} R_{1+z}\left(CR_{1+z}\right)^{2k+1}\left(CR_{1+z}\right)^{2N+2} \\
&= 2\sum_{k=0}^{N} R_{1+z}\left(CR_{1+z}\right)^{2k+1} + 2\sum_{k=0}^{\infty} R_{1+z}\left(CR_{1+z}\right)^{2k}\left(CR_{1+z}\right)^{2N+3}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(L^*L+z)^{-1} - (LL^*+z)^{-1} \\
&\quad = 2\sum_{k=0}^{N} R_{1+z}\left(CR_{1+z}\right)^{2k+1} + \Big( (L^*L+z)^{-1} - (LL^*+z)^{-1} \Big)\left(CR_{1+z}\right)^{2N+2} \\
&\quad = 2\sum_{k=0}^{N} R_{1+z}\left(CR_{1+z}\right)^{2k+1} + \Big( (L^*L+z)^{-1} + (LL^*+z)^{-1} \Big)\left(CR_{1+z}\right)^{2N+3},
\end{aligned}
$$

again by part $(i)$. Analytic continuation implies the asserted equalities. (The second term in item $(ii)$ is treated analogously). □

Before starting the proof that $\chi_\Lambda B_L(z)$, with $B_L(z)$ given by (7.2), is trace class, and then prove the trace formula in Theorem 7.1 for this operator, a closer inspection of the operators occuring in Proposition 7.4 with the help of Proposition 7.5 is in order. In particular, the principal aim of Lemma 7.7, is twofold: on one hand, we will prove that the power series representation of $B_L(z)$, basically derived in Proposition 7.5, starts with an operator essentially of the form

$$R_{1+z}\left(CR_{1+z}\right)^{2k+1}$$

for some $k \in \mathbb{N}_0$. For this kind of operators we have a trace class criterion at hand, Theorem 4.7 together with Corollary 4.3. On the other hand, we also prove representation formulas for the operators in (7.6) and (7.7). These formulas also start with operators involving high powers of $R_{1+z}$. This leads to continuity and differentiability properties for the corresponding integral kernels enabling the application of Proposition 5.5 and Lemma 5.6.

The key idea for proving Lemma 7.7, contained in Lemma 7.6, is to use the cancellation properties of the Euclidean Dirac algebra under the trace sign. For the Euclidean Dirac algebra we refer to Definition A.3; moreover, we refer to Proposition A.8 for the cancellation properties.

**Lemma 7.6.** *Let $L = \mathcal{Q} + \Phi$ be given by* (7.1)*. Let $z \in \mathbb{C}$ with* $\mathrm{Re}(z) > -1$ *and $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$ and recall $C = [Q, \Phi]$, $k \in \mathbb{N}$ odd. If either $k < n$ or $n$ is even, then*

$$\mathrm{tr}_{2^{\hat{n}}d}\left(R_{1+z}\big(CR_{1+z}\big)^k\right) = 0.$$

*Proof.* One observes using the fact that $\gamma_{j,n}$, $j \in \{1, \dots, n\}$, commutes with both $R_{1+z}$ and $(\partial_\ell \Phi)$, $\ell \in \{1, \dots, n\}$ (cf. Remark 6.1), that

$$R_{1+z}\big(CR_{1+z}\big)^k = R_{1+z}\Big(\sum_{\ell=1}^{n} \gamma_{\ell,n}(\partial_\ell \Phi) R_{1+z}\Big)^k$$

$$= R_{1+z} \sum_{\ell_1,\cdots,\ell_k=1}^{n} \gamma_{\ell_1,n}(\partial_{\ell_1}\Phi)R_{1+z}\cdots\gamma_{\ell_k,n}(\partial_{\ell_k}\Phi)R_{1+z}$$
$$= \sum_{\ell_1,\dots,\ell_k=1}^{n} \gamma_{\ell_1,n}\cdots\gamma_{\ell_k,n}R_{1+z}(\partial_{\ell_1}\Phi)R_{1+z}\cdots(\partial_{\ell_k}\Phi)R_{1+z}.$$

Next, employing

$$\begin{aligned}&\operatorname{tr}_{2^{\hat{n}}d}\Big(\gamma_{\ell_1,n}\cdots\gamma_{\ell_k,n}R_{1+z}(\partial_{\ell_1}\Phi)R_{1+z}\cdots(\partial_{\ell_k}\Phi)R_{1+z}\Big)\\ &\quad= \operatorname{tr}_{2^{\hat{n}}}\Big(\gamma_{\ell_1,n}\cdots\gamma_{\ell_k,n}\Big)\operatorname{tr}_d\Big(R_{1+z}(\partial_{\ell_1}\Phi)R_{1+z}\cdots(\partial_{\ell_k}\Phi)R_{1+z}\Big)\end{aligned}$$

for all $i_1,\dots,i_k \in \{1,\dots,n\}$, one concludes that

$$\operatorname{tr}_{2^{\hat{n}}d}\Big(R_{1+z}\big(CR_{1+z}\big)^k\Big) = 0,$$

by Proposition A.8. □

**Lemma 7.7.** *Let $L = \mathcal{Q} + \Phi$ be given by (7.1). Let $z \in \mathbb{C}$ with $\operatorname{Re}(z) > -1$ and $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$. One recalls $B_L(z)$, $J_L^j(z)$, and $A_L(z)$ given by (7.2), (7.6), and (7.7), respectively, as well as $R_{1+z}$ given by (4.6). Then the following assertions hold:*

$(i)$ *For all odd $n \in \mathbb{N}_{\geqslant 3}$,*

$$\begin{aligned}2B_L(z) &= \sum_{j=1}^{n}\big[\partial_j, J_L^j(z)\big] + A_L(z) \\ &= z\operatorname{tr}_{2^{\hat{n}}d}\big(2(R_{1+z}C)^nR_{1+z} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)(CR_{1+z})^{n+1}\big),\end{aligned} \tag{7.12}$$

*and, for all $j \in \{1,\dots,n\}$,*

$$\begin{aligned}J_L^j(z) &= 2\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\mathcal{Q}(R_{1+z}C)^{n-2}R_{1+z}\big) + 2\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\Phi(R_{1+z}C)^{n-1}R_{1+z}\big)\\ &\quad + \operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big)\\ &\quad + \operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\Phi\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big),\end{aligned}$$

*and*

$$\begin{aligned}A_L(z) &= \operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi, \Phi\big(2(R_{1+z}C)^nR_{1+z} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)\\ &\qquad\times(CR_{1+z})^{n+1}\big)\big]\big)\\ &\quad - \operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi, \mathcal{Q}\big(2(R_{1+z}C)^{n-1}R_{1+z} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)\\ &\qquad\times(CR_{1+z})^{n}\big)\big]\big).\end{aligned}$$

$(ii)$ *For all even $n \in \mathbb{N}$,*

$$B_L(z) = 0. \tag{7.13}$$

*Proof.* From Proposition 7.5, one has for $\operatorname{Re}(z) > -1$ and all $N \in \mathbb{N}$,

$$\begin{aligned}&(L^*L+z)^{-1} - (LL^*+z)^{-1}\\ &\quad= 2\sum_{k=0}^{N} R_{1+z}(CR_{1+z})^{2k+1} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)(CR_{1+z})^{2N+2}.\end{aligned}$$

In addition, using Lemma 7.6, one deduces that for $n$ even,

$$\operatorname{tr}_{2^{\hat{n}}d}\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big) = 0,$$

and, for $n$ odd,

$$\begin{aligned}
&\operatorname{tr}_{2^{\hat{n}}d}\big((L^*L+z)^{-1}-(LL^*+z)^{-1}\big)\\
&\quad=\operatorname{tr}_{2^{\hat{n}}d}\big(2(R_{1+z}C)^nR_{1+z}+\big((L^*L+z)^{-1}-(LL^*+z)^{-1}\big)(CR_{1+z})^{n+1}\big).
\end{aligned}$$

This proves (7.12).

In a similar fashion, using again Proposition 7.5 and the "cyclicity property" of $\operatorname{tr}_{2^{\hat{n}}d}$ (see Proposition 3.3), one obtains

$$\begin{aligned}
&\operatorname{tr}_{2^{\hat{n}}d}\big(L\,(L^*L+z)^{-1}\,\gamma_{j,n}\big)+\operatorname{tr}_{2^{\hat{n}}d}\big(L^*\,(LL^*+z)^{-1}\,\gamma_{j,n}\big)\\
&\quad=\operatorname{tr}_{2^{\hat{n}}d}\big(L\,(L^*L+z)^{-1}\,\gamma_{j,n}+L^*\,(LL^*+z)^{-1}\,\gamma_{j,n}\big)\\
&\quad=\operatorname{tr}_{2^{\hat{n}}d}\big(\,(\mathcal{Q}+\Phi)\,(L^*L+z)^{-1}\,\gamma_{j,n}+(-\mathcal{Q}+\Phi)\,(LL^*+z)^{-1}\,\gamma_{j,n}\big)\\
&\quad=\operatorname{tr}_{2^{\hat{n}}d}\big(\mathcal{Q}\,(L^*L+z)^{-1}\,\gamma_{j,n}-\mathcal{Q}\,(LL^*+z)^{-1}\,\gamma_{j,n}\big)\\
&\qquad+\operatorname{tr}_{2^{\hat{n}}d}\big(\Phi\big(\,(L^*L+z)^{-1}\,\gamma_{j,n}+(LL^*+z)^{-1}\,\gamma_{j,n}\big)\big)\\
&\quad=\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\big(\,(L^*L+z)^{-1}-(LL^*+z)^{-1}\,\big)\big)\\
&\qquad+\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\Phi\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\big)\\
&\quad=2\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\,(R_{1+z}C)^{n-2}\,R_{1+z}\big)+2\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\Phi\,(R_{1+z}C)^{n-1}\,R_{1+z}\big)\\
&\qquad+\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big)\\
&\qquad+\operatorname{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\Phi\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big),
\end{aligned}$$

and

$$\begin{aligned}
A_L(z)&=\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,L^*\,(LL^*+z)^{-1}\,\big]\big)-\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,L\,(L^*L+z)^{-1}\,\big]\big)\\
&=\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,L^*\,(LL^*+z)^{-1}-L\,(L^*L+z)^{-1}\,\big]\big)\\
&=\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,\Phi\,(LL^*+z)^{-1}-\Phi\,(L^*L+z)^{-1}\,\big]\big)\\
&\quad-\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,\mathcal{Q}\,(L^*L+z)^{-1}+\mathcal{Q}\,(LL^*+z)^{-1}\,\big]\big)\\
&=\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,\Phi\big(\,(LL^*+z)^{-1}-(L^*L+z)^{-1}\,\big)\big]\big)\\
&\quad-\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,\mathcal{Q}\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\big]\big)\\
&=\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,\Phi\big(2\,(R_{1+z}C)^n\,R_{1+z}+\big(\,(L^*L+z)^{-1}-(LL^*+z)^{-1}\,\big)\\
&\qquad\qquad\times(CR_{1+z})^{n+1}\,\big)\big]\big)\\
&\quad-\operatorname{tr}_{2^{\hat{n}}d}\big(\big[\Phi,\mathcal{Q}\big(2\,(R_{1+z}C)^{n-1}\,R_{1+z}+\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\\
&\qquad\qquad\times(CR_{1+z})^n\,\big)\big]\big).
\end{aligned}$$

□

One important upshot of Lemma 7.7 is the fact (7.13), implying that only odd dimensions are of interest when computing the index of $L$. Thus, we will focus on the case $n$ odd, only.

The next theorem concludes this section and asserts that the trace class assumptions on $B_L(z)$ in Theorem 3.4 are satisfied for $B_L(z)$ given by (7.2). As the sequence $\{T_\Lambda\}_\Lambda$ we shall use $\{\chi_\Lambda\}_\Lambda$ the sequence of multiplication operators induced by multiplying with the cut-off (characteristic) function $\chi_\Lambda$. The sequence $\{S^*_\Lambda\}_\Lambda$ is set to be the constant sequence $S_\Lambda=I_{L^2(\mathbb{R}^n)}$ for all $\Lambda$. Clearly, $\chi_\Lambda\in L^{n+1}(\mathbb{R}^n)$ for all $\Lambda>0$.

**Theorem 7.8.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $L = \mathcal{Q} + \Phi$ given by* (7.1)*. Then there exists $\delta > 0$ such that for all $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$ and $\Lambda > 0$, the operator $\chi_\Lambda B_L(z)$ with $B_L(z)$ given by* (7.2) *is trace class with $z \mapsto \operatorname{tr}(|\chi_\Lambda B_L(z)|)$ bounded on $B(0,\delta)\backslash\{0\}$.*

*Proof.* We start by showing that $z \mapsto \chi_\Lambda R_{1+z}\big((CR_{1+z})^n\big)$ is trace class with trace class norm being bounded around a neighborhood of 0, where $C = (\mathcal{Q}\Phi) = \sum_{j=1}^n \gamma_{j,n}\,(\partial_j \Phi)$ is given by (6.15), see also Remark 2.1, and $R_{1+z}$ is given by (4.6). Using $n = 2\widehat{n} + 1$ we write

$$\chi_\Lambda R_{1+z}\big(CR_{1+z})^n\big) = \Big(\chi_\Lambda R_{1+z}\,(CR_{1+z})^{\widehat{n}}\Big)\,\Big((CR_{1+z})^{\widehat{n}+1}\Big).$$

By Theorem 4.7 the operators

$$\Big(\chi_\Lambda R_{1+z}\,(CR_{1+z})^{\widehat{n}}\Big) \text{ and } \Big((CR_{1+z})^{\widehat{n}+1}\Big)$$

are Hilbert–Schmidt by the admissability of $\Phi$ (in this context, see, in particular, Definition 6.11 $(iii)$). Moreover, the boundedness of $z \mapsto \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\,(R_{1+z}C)^n$ with respect to the norm in $\mathcal{B}_1\left(L^2(\mathbb{R}^n)\right)$ around a neighborhood of 0, now follows from Theorem 4.2 together with the estimates in Theorem 4.7 and Lemma 4.5 (we note that we apply these statements for $\mu = 1 + z$ with $z \in \mathbb{C}_{\mathrm{Re}>-1}$).

One recalls (employing the spectral theorem) that for all self-adjoint operators $A$ on a Hilbert space $\mathcal{H}$ with 0 being an isolated eigenvalue, the operator family $z \mapsto z\,(A+z)^{-1}$ is uniformly bounded on $B(0,\delta)$ for some $\delta > 0$. By Lemma 7.7, (7.12), the uniform boundedness of $z \mapsto z\,(A+z)^{-1}$ on $B(0,\delta)$ for some $\delta > 0$, and the ideal property for trace class operators, it remains to show that $(CR_{1+z})^{n+1}$ is trace class, with trace class norm bounded for $z \in B(0,\delta')$ for some sufficiently small $\delta' > 0$. For $n = 2\widehat{n} + 1$, one observes that $(CR_{1+z})^{n+1}$ is a sum of operators of the form

$$\Psi_1 \cdots R_{1+z}\Psi_{n+1}R_{1+z} = (\Psi_1 \cdots R_{1+z}\Psi_{\widehat{n}+1}R_{1+z})\,(\Psi_{\widehat{n}+2} \cdots R_{1+z}\Psi_{2\widehat{n}+2}R_{1+z}),$$

where $\Psi_j$ are multiplication operators with bounded $C^\infty$-functions with the property that for some constant $\kappa > 0$, $|\Psi_j(x)| \leqslant \frac{\kappa}{1+|x|}$, $x \in \mathbb{R}^n$. For deriving the trace class property of

$$\Psi_1 \cdots R_{1+z}\Psi_{n+1}R_{1+z} = (\Psi_1 \cdots R_{1+z}\Psi_{\widehat{n}+1}R_{1+z})\,(\Psi_{\widehat{n}+2} \cdots R_{1+z}\Psi_{2\widehat{n}+2}R_{1+z}),$$

we use Theorem 4.7 and Lemma 4.5. Let $z_0 \in (-1,0)$. By Theorem 4.7 $(i)$ one estimates for some $\kappa' > 0$, depending on $\widehat{n}$, $\kappa$ and $z_0$, and all $z \in \mathbb{C}_{\geqslant z_0}$,

$$\|\,(\Psi_1 \cdots R_{1+z}\Psi_{\widehat{n}+1}R_{1+z})\,\|_{\mathcal{B}_2} \leqslant \prod_{j=1}^{\widehat{n}+1} \|\Psi_j R_{1+z}\| \leqslant \kappa' \prod_{j=1}^{\widehat{n}+1} \|\Psi_j\|_{L^{n+1}},$$

where we used Lemma 4.5 in the last estimate. The same argument applies to

$$(\Psi_{\widehat{n}+2} \cdots R_{1+z}\Psi_{2\widehat{n}+2}R_{1+z}).$$

This concludes the proof since $(CR_{1+z})^{n+1}$ is trace class by Theorem 4.2. $\square$

*Remark* 7.9. We note that the method of proof of Theorem 7.8 shows that the trace of $\chi_\Lambda B_L(z)$ can be computed as the integral over the diagonal of the respective integral kernel. In fact, we have shown that $\chi_\Lambda B_L(z)$ may be represented as sums of products of two Hilbert–Schmidt operators leading to the trace formula given in Corollary 4.3. $\diamond$

## 8. Derivation of the Trace Formula – Diagonal Estimates

In this section, we shall compute the trace of $\chi_\Lambda B_L(z)$, $\Lambda > 0$, $z \in \varrho(-L^*L) \cap \varrho(-LL^*) \cap \mathbb{C}_{\mathrm{Re}>-1}$, with $B_L$ given by (3.2). After stating the next lemma (needed to be able to apply Lemma 5.6 and Proposition 5.5 to the sum in (7.5)) we will outline the strategy of the proof.

We note that for the application of Lemma 5.6 to the first summand in (7.5), one needs to establish continuous differentiability of the integral kernel of (7.6). In this context we emphasize the different regularity of the kernels of (7.6) for $n = 3$ and $n \geqslant 5$, necessitating modifications for the case $n = 3$ due to the lack of differentiability of (7.6).

**Lemma 8.1** ([22, Lemma 4, p. 224])**.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $L = \mathcal{Q} + \Phi$ be given by (7.1), and let $z \in \varrho(-LL^*) \cap \varrho(-L^*L)$, with $\mathrm{Re}(z) > -1$. Denote the integral kernels of the following operators*

$$J_L^j(z) = \mathrm{tr}_{2^{\widehat{n}}d}\big(L\,(L^*L+z)^{-1}\,\gamma_{j,n}\big) - \mathrm{tr}_{2^{\widehat{n}}d}\big(L^*\,(LL^*+z)^{-1}\,\gamma_{j,n}\big),$$
$$A_L(z) = \mathrm{tr}_{2^{\widehat{n}}d}\big(\big[\Phi, L^*\,(LL^*+z)^{-1}\big]\big) - \mathrm{tr}_{2^{\widehat{n}}d}\big(\big[\Phi, L\,(L^*L+z)^{-1}\big]\big),$$

*by $G_{J,j,z}$, $j \in \{1,\dots,n\}$, and $G_{A,z}$, respectively. Then $G_{A,z}$ is continuous and satisfies $G_{A,z}(x,y) \to 0$ if $y \to x$ for all $x \in \mathbb{R}^n$. If $n \geqslant 5$, $G_{J,j,z}$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$.*

*Proof.* Appealing to Lemma 7.7, one recalls with $R_{1+z}$, $\mathcal{Q}$, and $C$ given by (4.6), (6.3), and (6.15), respectively,

$$\begin{aligned} J_L^j(z) &= 2\,\mathrm{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\,(R_{1+z}C)^{n-2}\,R_{1+z}\big) + 2\,\mathrm{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\Phi\,(R_{1+z}C)^{n-1}\,R_{1+z}\big) \\ &\quad + \mathrm{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\big(\,(L^*L+z)^{-1} + (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big) \\ &\quad + \mathrm{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\Phi\big(\,(L^*L+z)^{-1} + (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big), \quad j \in \{1,\dots,n\}, \end{aligned}$$

and

$$\begin{aligned} A_L(z) = \mathrm{tr}_{2^{\widehat{n}}d}\big(\big[\Phi, \Phi\big(2\,(R_{1+z}C)^n\,R_{1+z} &+ \big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big) \\ &\times (CR_{1+z})^{n+1}\,\big)\big]\big) \\ - \mathrm{tr}_{2^{\widehat{n}}d}\big(\big[\Phi, \mathcal{Q}\big(2\,(R_{1+z}C)^{n-1}\,R_{1+z} &+ \big(\,(L^*L+z)^{-1} + (LL^*+z)^{-1}\,\big) \\ &\times (CR_{1+z})^n\,\big)\big]\big). \end{aligned}$$

By Proposition 5.4 (one recalls that $\Phi$ is admissible and hence $\Phi \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ by Definition 6.11 $(i)$), one gets for all $\ell \in \mathbb{R}$,

$$Q\gamma_{j,n}\,(R_{1+z}C)^{n-2}\,R_{1+z} \in \mathcal{B}\big(H^\ell(\mathbb{R}^n)^{2^{\widehat{n}}d}, H^{\ell+2(n-2)+2-1}(\mathbb{R}^n)^{2^{\widehat{n}}d}\big).$$

For $n \geqslant 5$, one obtains from $(2(n-2)+2-1) = n-3 > 0$, the continuity of $G_{J,j,z}$ by Corollary 5.3. Moreover, since $(2(n-2)+2-1) - n - 1 = n - 4 > 0$, Corollary 5.3 also implies continuous differentiability of $G_{J,j,z}$. Similar arguments ensure the continuity of the integral kernel of $A_L(z)$ (for $n \geqslant 3$). Moreover, for $n \geqslant 3$, the integral kernel of $A_L(z)$ vanishes on the diagonal by Proposition 5.5. $\square$

Next, we outline the idea for computing the trace of $\chi_\Lambda B_L(z)$. By Theorem 3.4 and Theorem 7.1, we know that the limit $\lim_{\Lambda\to\infty} \mathrm{tr}(\chi_\Lambda B_L(0))$ exists. However, in order to derive the explicit formula asserted in Theorem 7.1 also for $z$ in a neighborhood of 0, some work is required. As it will turn out, for $z$ with large real

part – at least for a sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ – we can show that an expression similar to the one in Theorem 7.1 is valid.

For achieving the existence of the limit (without using sequences) for $z$ in a neighborhood of 0, we intend to employ Montel's theorem. One recalls that for an open set $U \subseteq \mathbb{C}$, a set $\mathcal{G} \subseteq \mathbb{C}^U := \{f \,|\, f\colon U \to \mathbb{C}\}$ is called *locally bounded*, if for all compact $\Omega \subset U$,

$$\sup_{f\in\mathcal{G}} \sup_{z\in\Omega} |f(z)| < \infty. \tag{8.1}$$

**Theorem 8.2** (Montel's theorem, see, e.g., [35], p. 146–154)**.** *Let $U \subseteq \mathbb{C}$ open, $\{f_\Lambda\}_{\Lambda\in\mathbb{N}}$ a locally bounded family of analytic functions on $U$. Then there exists a subsequence $\{f_{\Lambda_k}\}_{k\in\mathbb{N}}$ and an analytic function $g$ on $U$ such that $f_{\Lambda_k} \to g$ as $k \to \infty$ in the compact open topology (i.e., for any compact set $\Omega \subset U$, the sequence $\{f_{\Lambda_k}|_\Omega\}_{k\in\mathbb{N}}$ converges uniformly to $g|_\Omega$).*

For our particular application of Montel's theorem, we need to show that the family of analytic functions

$$\{z \mapsto \operatorname{tr}(\chi_\Lambda B_L(z))\}_\Lambda$$

constitutes a locally bounded family. Thus, one needs to show that for all compact $\Omega \subset \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$,

$$\sup_{\Lambda>0} \sup_{z\in\Omega} |\operatorname{tr}(\chi_\Lambda B_L(z))| < \infty. \tag{8.2}$$

For this assertion, it is crucial that some integral kernels involved in the computation of the trace vanish on the diagonal, see, for instance, Proposition 5.5. We note that generally, the expression

$$\sup_{\Lambda>0} \sup_{z\in\Omega} \operatorname{tr}(|(\chi_\Lambda B_L(z))|), \tag{8.3}$$

cannot be finite, as the example constructed in Appendix B demonstrates. In order to prove (8.2), we actually show for all $\Omega \subset \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$ compact,

$$\sup_{\Lambda>0} \sup_{z\in\Omega} |z \operatorname{tr}(\chi_\Lambda B_L(z))| < \infty, \tag{8.4}$$

and then appeal to the fact that condition (8.4) together with Theorem 7.8 implies (8.2), as the next result confirms:

**Lemma 8.3.** *Assume that $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence of analytic (scalar-valued) functions on $B_{\mathbb{C}}(0,1)$. Assume that $\{z \mapsto z\phi_k(z)\}_{k\in\mathbb{N}}$ is locally bounded on $B(0,1)$. Then $\{\phi_k\}_{k\in\mathbb{N}}$ is locally bounded on $B(0,1)$.*

*Proof.* Assume that $\{\phi_k\}_{k\in\mathbb{N}}$ is not locally bounded on $B(0,1)$. Then there exists a subsequence $\{\phi_{k_\ell}\}_{\ell\in\mathbb{N}}$ and a corresponding sequence of complex numbers $\{z_{k_\ell}\}_{\ell\in\mathbb{N}}$ with the property that $z_{k_\ell} \to 0$ and $|\phi_{k_\ell}(z_{k_\ell})| \to \infty$ as $\ell \to \infty$. Since

$$\{\psi_\ell\}_{\ell\in\mathbb{N}} := \{z \mapsto z\phi_{k_\ell}(z)\}_{\ell\in\mathbb{N}}$$

is locally bounded on $B(0,1)$ there exists an accumulation point $\psi$ in the compact open topology of analytic functions $\mathcal{H}(B(0,1))$ on $B(0,1)$ by Montel's theorem. Without loss of generality, one can assume that $\psi_\ell \to \psi$ in $\mathcal{H}(B(0,1))$ as $\ell \to \infty$. By construction, one has $\psi_\ell(0) = 0$ and for some $r > 0$,

$$\left|\frac{1}{z}\psi_\ell(z) - \psi'(0)\right| \leqslant \left|\frac{1}{z}\left(\psi_\ell(z) - \psi_\ell(0)\right) - \psi_\ell'(0)\right| + |\psi_\ell'(0) - \psi'(0)|$$

$$\leqslant \sup_{z\in B(0,r)} |(\psi_\ell'(z)-\psi_\ell'(0))| + |\psi_\ell'(0)-\psi'(0)|$$

for all $z \in B(0,r)\backslash\{0\}$. Since $\psi_\ell' \to \psi'$ uniformly on compacts, it follows that

$$\limsup_{\ell\to\infty} \sup_{z\in B(0,r)\backslash\{0\}} \left|\frac{1}{z}\psi_\ell(z)\right| < \infty.$$

However, for $\ell$ sufficiently large, one concludes

$$\sup_{z\in B(0,r)\backslash\{0\}} \left|\frac{1}{z}\psi_\ell(z)\right| \geqslant \left|\frac{1}{z_{k_\ell}}\psi_\ell(z_{k_\ell})\right| = |\phi_{k_\ell}(z_{k_\ell})| \underset{\ell\to\infty}{\longrightarrow} \infty,$$

a contradiction. ☐

*Remark* 8.4. It turns out that the analyticity hypothesis in Lemma 8.3 is crucial. Indeed, for every $n \in \mathbb{N}$, there exists a $C^\infty$-function $\psi_n \colon [0,1) \to [0,\infty)$ with the properties,

$$\psi_n|_{(0,1/(2n))} = 0, \quad 0 \leqslant \psi_n(x) \leqslant \psi_n\left(\frac{1}{n}\right) = n, \quad \psi_n|_{(2/n,1)} = 0.$$

Then $\psi_n(0) = 0$ and $0 \leqslant x\psi_n(x) \leqslant (2/n)n = 2$. Considering $\phi_n(x+iy) \coloneqq \psi_n(|x+iy|)$ for $x,y\in\mathbb{R}$, $x+iy \in B(0,1)$, $n\in\mathbb{N}$, one gets that $\phi_n$ is real differentiable and the assumptions of Lemma 8.3, except for analyticity, are all satisfied. In addition, $\phi_n(0) = 0$, however, $\phi_n(1/n) = n \to \infty$ as $n \to \infty$. Thus, $\{\phi_n\}_{n\in\mathbb{N}}$ is not locally bounded on $B(0,1)$. ⋄

The next aim of this section is to establish Theorem 8.7, that is, an important step for obtaining (8.2). The terms to be discussed in Theorem 8.7 split up into a leading order term and the rest. The first term will be studied in Lemma 8.5 and the second one in Lemma 8.6. The strategy of proof in these lemmas is the same. It rests on the following observation: Let $U \subseteq \mathbb{C}$ open, $U \ni z \mapsto T(z) \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big)$. Assume that for all $z \in U$ we have $T(z) \in \mathcal{B}_1\big(L^2(\mathbb{R}^n)\big)$ and that $z \mapsto \operatorname{tr}(|T(z)|)$ is locally bounded. Then

$$\{z \mapsto \operatorname{tr}(\chi_\Lambda T(z))\}_{\Lambda>0}$$

is locally bounded as well. Indeed, the assertion follows from the boundedness of the family $\{\chi_\Lambda\}_{\Lambda>0}$ as bounded linear (multiplication) operators in $\mathcal{B}(L^2(\mathbb{R}^n))$ and the ideal property of the trace class. In the situations to be considered in the following, the trace class property for $T(z)$ will be shown with the help of the results of Section 4.

**Lemma 8.5.** *Let $L = \mathcal{Q} + \Phi$ be given by* (7.1) *and for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > -1$ let $R_{1+z}$ be given by* (4.6) *and $C$ as in* (6.15)*, $n \in \mathbb{N}_{>1}$ odd. For $j \in \{1,\dots,n\}$, let $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}$ as in Remark* 6.1 *and $\chi_\Lambda$ as in* (7.3)*, $\Lambda > 0$. For $z \in \mathbb{C}_{\operatorname{Re}>-1}$ consider*

$$\psi_\Lambda(z) \coloneqq \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big([\mathcal{Q}, \Phi\,(CR_{1+z})^n]\big)$$

*and*

$$\widetilde{\psi}_\Lambda(z) \coloneqq \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big([\mathcal{Q}, \mathcal{Q}\,(CR_{1+z})^n]\big).$$

*Then for all $z \in \mathbb{C}_{\operatorname{Re}>-1}$, the operators $\psi_\Lambda(z)$, $\widetilde{\psi}_\Lambda(z)$ are trace class and the families*

$$\{z \mapsto \operatorname{tr}_{L^2(\mathbb{R}^n)}(\psi_\Lambda(z))\}_{\Lambda>0} \text{ and } \big\{z \mapsto \operatorname{tr}_{L^2(\mathbb{R}^n)}\big(\widetilde{\psi}_\Lambda(z)\big)\big\}_{\Lambda>0}$$

*are locally bounded* (*cf.* (8.1))*.*

*Proof.* First we deal with $\psi_\Lambda(z)$. One computes,

$$\psi_\Lambda(z) = \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big([\mathcal{Q}, \Phi\,(CR_{1+z})^n]\big) = \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(\mathcal{Q}\Phi\,(CR_{1+z})^n - \Phi\,(CR_{1+z})^n\,\mathcal{Q}\big).$$

Before we discuss the latter operator, we note that

$$\begin{aligned}\mathcal{Q}\Phi\,(CR_{1+z})^n &= \Phi\mathcal{Q}\,(CR_{1+z})^n + [\mathcal{Q},\Phi]\,(CR_{1+z})^n \\ &= \Phi\bigg(\sum_{j=1}^n (CR_{1+z})^{j-1}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j} + (CR_{1+z})^n\mathcal{Q}\bigg) \\ &\quad + [\mathcal{Q},\Phi]\,(CR_{1+z})^n\,,\end{aligned}$$

where the latter equality follows via an induction argument. Hence,

$$\begin{aligned}\psi_\Lambda(z) = \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\Bigg(&\Phi\bigg(\sum_{j=1}^n (CR_{1+z})^{j-1}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j}\bigg) \\ &+ [\mathcal{Q},\Phi](CR_{1+z})^n\Bigg).\end{aligned} \tag{8.5}$$

Next, with the results of Section 4, we will deduce that the operator family

$$z \mapsto \Bigg(\Phi\bigg(\sum_{j=1}^n (CR_{1+z})^{j-1}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j}\bigg) + [\mathcal{Q},\Phi]\,(CR_{1+z})^n\Bigg) \tag{8.6}$$

is trace class, which – together with the estimates in Lemma 4.5 – establishes the assertion for $\psi_\Lambda$: Indeed, the only difference between (8.5) and (8.6) is the prefactor $\chi_\Lambda$. So we get the assertion with the help of the reasoning prior to Lemma 8.5. In order to observe that each summand in (8.6) is trace class, we proceed as follows. Recall $n = 2\widehat{n}+1$ and let $j \in \{1,\dots,\widehat{n}\}$ (the case $n-j \in \{1,\dots,\widehat{n}\}$ can be dealt with similarly). Then, by the admissability of $\Phi$ (see Hypothesis 6.11), one infers that $[\mathcal{Q},C]$ is a multiplication operator with

$$|[\mathcal{Q},C](x)| \leqslant \kappa(1+|x|)^{-1-\varepsilon}, \quad x \in \mathbb{R}^n.$$

Hence, as $1+\varepsilon > 3/2$ by Definition 6.11, Theorem 4.9 applies and guarentees that

$$(CR_{1+z})^{j-1}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{\widehat{n}-j}$$

is Hilbert–Schmidt. Using Theorem 4.7, one deduces that $(CR_{1+z})^{\widehat{n}+1}$ is also Hilbert–Schmidt and thus

$$(CR_{1+z})^{j-1}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{\widehat{n}-j}(CR_{1+z})^{\widehat{n}+1}$$

is trace class, by Theorem 4.2.

For $\widetilde{\psi}_\Lambda$ one proceeds similarly. First one notes that

$$\widetilde{\psi}_\Lambda(z) = \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\Bigg(\mathcal{Q}\bigg(\sum_{j=1}^n (CR_{1+z})^{j-1}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j}\bigg)\Bigg). \tag{8.7}$$

Applying Theorems 4.7, 4.9, and 4.2, one infers the assertion for $\widetilde{\psi}_\Lambda$. However, one has to use the respective assertions, where some of the resolvents of the Laplacian is replaced by $\mathcal{Q}$ times the resolvents. Indeed, in the sum in (8.7), the term for $j=1$ yields

$$\begin{aligned}\mathcal{Q}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-1} &= [\mathcal{Q},[\mathcal{Q},C]]R_{1+z}(CR_{1+z})^{n-1} \\ &\quad + [\mathcal{Q},C]\mathcal{Q}R_{1+z}(CR_{1+z})^{n-1}\end{aligned}$$

$$= [\mathcal{Q},[\mathcal{Q},C]]R_{1+z}(CR_{1+z})^{n-1} + [\mathcal{Q},C]R_{1+z}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-2} \\ + [\mathcal{Q},C]R_{1+z}C\mathcal{Q}R_{1+z}(CR_{1+z})^{n-2},$$

and for $j' \in \{2,\ldots,n\}$ one obtains

$$\mathcal{Q}CR_{1+z}(CR_{1+z})^{j'-2}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j'} \\ = [\mathcal{Q},C]R_{1+z}(CR_{1+z})^{j'-2}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j'} \\ + C\mathcal{Q}R_{1+z}(CR_{1+z})^{j'-2}[\mathcal{Q},C]R_{1+z}(CR_{1+z})^{n-j'}. \qquad \square$$

The next lemma is the reason, why we have to invoke Lemma 8.3 in our argument. The crucial point is that we can use the Neumann series expressions for the resolvents $(L^*L+z)^{-1}$ and $(LL^*+z)^{-1}$ only for $z$ with large real part. But for $z$ in the vicinity of 0, we do not have such a representation. Using again the ideal property for trace class operators, we can, however, bound $z(L^*L+z)^{-1}$ for small $z$ in the $\mathcal{B}(L^2(\mathbb{R}^n))$-norm. Introducing the sector $\Sigma_{z_0,\vartheta} \subset \mathbb{C}$ by

$$\Sigma_{z_0,\vartheta} := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > z_0, |\arg(\mu)| < \vartheta\}, \tag{8.8}$$

for some $z_0 \in \mathbb{R}$ and $\vartheta \in (0,\pi/2)$, the result reads as follows:

**Lemma 8.6.** *Let $L = \mathcal{Q} + \Phi$ be given by* (7.1) *and for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > -1$, let $R_{1+z}$ be given by* (4.6) *and $C$ as in* (6.15)*, $\chi_\Lambda$ as in* (7.3)*, $\Lambda > 0$. For $j \in \{1,\ldots,n\}$, let $\gamma_{j,n} \in \mathbb{C}^{2^{\hat{n}}\times 2^{\hat{n}}}$ as in Remark* 6.1*. For $z \in \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$ consider*

$$\eta_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\left(\left[\mathcal{Q}, \Phi\left(\left(L^*L+z\right)^{-1} - \left(LL^*+z\right)^{-1}\right)\left(CR_{1+z}\right)^{n+1}\right]\right)$$

*and*

$$\widetilde{\eta}_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\left(\left[\mathcal{Q}, \left(\mathcal{Q}\left(\left(L^*L+z\right)^{-1} - \left(LL^*+z\right)^{-1}\right)\left(CR_{1+z}\right)^{n+1}\right)\right]\right).$$

*Then for all $z \in \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$, the operators $\eta_\Lambda(z)$, $\widetilde{\eta}_\Lambda(z)$ are trace class. There exists $\delta \in (-1,0)$, $\vartheta \in (0,\pi/2)$ such that the families*

$$\{\Sigma_{\delta,\vartheta} \cup \mathbb{C}_{\mathrm{Re}>0} \ni z \mapsto z \operatorname{tr}_{L^2(\mathbb{R}^n)}(\eta_\Lambda(z))\}_{\Lambda>0}$$

*and*

$$\left\{\Sigma_{\delta,\vartheta} \cup \mathbb{C}_{\mathrm{Re}>0} \ni z \mapsto z \operatorname{tr}_{L^2(\mathbb{R}^n)}\left(\widetilde{\eta}_\Lambda(z)\right)\right\}_{\Lambda>0}$$

*are locally bounded* (*cf.* (8.1))*.*

*Proof.* By the Fredholm property of $L$ there exist $\delta \in (-1,0)$ and $\vartheta \in (0,\pi/2)$ such that

$$\Sigma_{\delta,\vartheta}\backslash\{0\} \ni z \mapsto z\left(L^*L+z\right)^{-1} \text{ and } \Sigma_{\delta,\vartheta}\backslash\{0\} \ni z \mapsto z\left(LL^*+z\right)^{-1}$$

have analytic extensions to $\Sigma_{\delta,\vartheta}$. Let $\Omega \subset \Sigma_{\delta,\vartheta} \cup \mathbb{C}_{\mathrm{Re}>0}$ be compact. One notes that

$$\Omega \ni z \mapsto z\left(L^*L+z\right)^{-1} \text{ and } \Omega \ni z \mapsto z\left(LL^*+z\right)^{-1}$$

define bounded families of bounded linear operators from $L^2(\mathbb{R}^n)^{2^{\hat{n}}d}$ to $H^2(\mathbb{R}^n)^{2^{\hat{n}}d}$. Indeed, by Proposition 6.10, one infers that $\phi \mapsto \|(L^*L+1)\phi\|$ and $\phi \mapsto \|(LL^*+1)\phi\|$ define equivalent norms on $H^2(\mathbb{R}^n)^{2^{\hat{n}}d}$. Hence, for $\phi \in L^2(\mathbb{R}^n)^{2^{\hat{n}}d}$ and $z \in \Omega\backslash\{0\}$ one computes

$$\left\|(L^*L+1)z(L^*L+z)^{-1}\phi\right\| = |z|\left\|(L^*L+z+1-z)(L^*L+z)^{-1}\phi\right\| \\ \leqslant |z|\|\phi\| + |z||(1-z)|\frac{1}{|z|}\|\phi\|.$$

Next, consider

$$\begin{aligned}\eta_\Lambda(z) &= \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(\big[\mathcal{Q}, \Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,\big]\big)\\ &= \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(\mathcal{Q}\Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\\ &\qquad - \Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,\mathcal{Q}\big).\end{aligned}$$

For the first summand one observes that

$$\begin{aligned}&\mathcal{Q}\Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\\ &\quad = C\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\\ &\qquad + \Phi\mathcal{Q}\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,.\end{aligned}$$

Employing our observation at the beginning of the proof and Theorem 6.4, one realizes that

$$\Omega \ni z \mapsto \mathcal{Q}z\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)$$

defines a bounded family of bounded linear operators in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$. Thus, since $\Omega \ni z \mapsto (CR_{1+z})^{n+1}$ is a family of trace class operators,

$$\begin{aligned}\Omega \ni z \mapsto & z\chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(\mathcal{Q}\Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,\big)\\ &= \operatorname{tr}_{2^{\widehat{n}}d}\big(z\chi_\Lambda\big(\mathcal{Q}\Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,\big)\big)\end{aligned}$$

is uniformly bounded in $\mathcal{B}_1$, with bound independently of $\Lambda > 0$, upon appealing to the ideal property of trace class operators.

The second summand requires the observation that

$$(CR_{1+z})^{n+1}\,\mathcal{Q} = (CR_{1+z})^{n}\,(CR_{1+z})\,\mathcal{Q} = (CR_{1+z})^{n}\,(C\mathcal{Q}R_{1+z})$$

defines a bounded family of trace class operators for $z \in \Omega$, proving the assertion for $\eta_\Lambda$.

The corresponding assertion for $\widetilde{\eta}_\Lambda$ is conceptually the same. In fact, it follows from the observation that

$$\begin{aligned}\Omega \ni z \mapsto & \mathcal{Q}\mathcal{Q}z\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)\\ &= \Delta I_{2^{\widehat{n}}d}\, z\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)\end{aligned}$$

is a bounded family of bounded linear operators by our preliminary observation that $\Omega \ni z \to z(L^*L+z)^{-1}$ and $\Omega \ni z \to z(LL^*+z)^{-1}$ define uniformly bounded operator families from $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ to $H^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$, as well as using again the fact that

$$\Omega \ni z \mapsto (CR_{1+z})^{n+1}\,\mathcal{Q} \text{ and } \Omega \ni z \mapsto (CR_{1+z})^{n+1}$$

constitute bounded families of trace class operators. $\square$

Lemmas 8.5 and 8.6 can be summarized as follows.

**Theorem 8.7.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, let $L = \mathcal{Q} + \Phi$ be given by* (7.1) *and for $z \in \mathbb{C}$ with* $\operatorname{Re}(z) > -1$ *let $R_{1+z}$ be given by* (4.6) *and $C$ as in* (6.15). *For $j \in \{1, \dots, n\}$, let $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}} \times 2^{\widehat{n}}}$ as in Remark* 6.1. *For $z \in \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-LL^*) \cap \varrho(L^*L)$, introduce*

$$\phi_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(\big[\mathcal{Q}, \Phi\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big]\big)$$

*and*

$$\widetilde{\phi}_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(\big[\mathcal{Q}, \big(\mathcal{Q}\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big)\big]\big).$$

*Then for all* $z \in \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-LL^*) \cap \varrho(L^*L)$, *the operators* $\phi_\Lambda(z)$, $\widetilde{\phi}_\Lambda(z)$ *are trace class. There exists* $\delta \in (-1,0)$, $\vartheta \in (0,\pi/2)$ *such that the families*

$$\{\Sigma_{\delta,\vartheta} \cup \mathbb{C}_{\mathrm{Re}>0} \ni z \mapsto \operatorname{tr}_{L^2(\mathbb{R}^n)}(z\phi_\Lambda(z))\}_{\Lambda>0}$$

*and*

$$\big\{\Sigma_{\delta,\vartheta} \cup \mathbb{C}_{\mathrm{Re}>0} \ni z \mapsto \operatorname{tr}_{L^2(\mathbb{R}^n)}\big(z\widetilde{\phi}_\Lambda(z)\big)\big\}_{\Lambda>0}$$

*are locally bounded* (*cf.* (8.1)).

*Proof.* One recalls from equations (7.8) and (7.9) the expressions

$$\begin{aligned}(L^*L+z)^{-1} &= I + (L^*L+z)^{-1}\, CR_{1+z},\\ (LL^*+z)^{-1} &= I - (LL^*+z)^{-1}\, CR_{1+z}.\end{aligned}$$

Hence, one gets

$$\begin{aligned}\phi_\Lambda(z) &= \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(2\,[\mathcal{Q}, \Phi\,(CR_{1+z})^n]\,\big)\\ &\quad + \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\Big(\big[\mathcal{Q}, \Phi\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,\big]\Big)\\ &= 2\psi_\Lambda(z) + \eta_\Lambda(z),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\phi}_\Lambda(z) &= \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big(2\,[\mathcal{Q}, \mathcal{Q}\,(CR_{1+z})^n]\,\big)\\ &\quad + \chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\Big(\big[\mathcal{Q}, \mathcal{Q}\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n+1}\,\big]\Big)\\ &= 2\widetilde{\psi}_\Lambda(z) + \widetilde{\eta}_\Lambda(z),\end{aligned}$$

with the functions introduced in Lemmas 8.5 and 8.6. Thus, the assertion on the local boundedness follows from these two lemmas. $\square$

The forthcoming statements are used for showing that for computing the trace the only term that matters is discussed in Proposition 8.13. We recall that by Remark 7.9, one can compute the trace of $\chi_\Lambda B_L(z)$ as the integral over the diagonal of its integral kernel. So the estimates on the diagonal derived in Section 5 will be used in the following. We shall elaborate on this idea further after having stated the next two auxiliaury results. Both these results serve to show that some integral kernels actually vanish on the diagonal.

**Lemma 8.8.** *Let* $n \in \mathbb{N}_{\geqslant 3}$ *be odd,* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > -1$. *Let* $R_{1+z}$, $\mathcal{Q}$, $C$, *and* $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}} \times 2^{\widehat{n}}}$, $j \in \{1,\dots,n\}$, *be given by* (4.6), (6.3), (6.15) *and as in Remark* 6.1, *respectively. Let* $\Phi\colon \mathbb{R}^n \to \mathbb{C}^{d\times d}$ *be admissible* (*see Definition* 6.11). *Then for all* $j \in \{1,\dots,n\}$,

$$\operatorname{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\,(R_{1+z}C)^{n-2}\,R_{1+z}\big) = -\operatorname{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\mathcal{Q}\,(R_{1+z}\Phi\mathcal{Q})^{n-2}\,R_{1+z}\big). \tag{8.9}$$

*Proof.* One has

$$\begin{aligned}\operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}R_{1+z}CR_{1+z}\right) &= \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}R_{1+z}\left(\mathcal{Q}\Phi - \Phi\mathcal{Q}\right)R_{1+z}\right),\\ &= \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}R_{1+z}\mathcal{Q}\mathcal{Q}\Phi R_{1+z}\right)\\ &\quad - \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}R_{1+z}\Phi\mathcal{Q}R_{1+z}\right)\\ &= -\operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}R_{1+z}\Phi\mathcal{Q}R_{1+z}\right),\end{aligned}$$

using Proposition A.8 to deduce that $\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}R_{1+z}\mathcal{Q}\mathcal{Q}\Phi R_{1+z}\right)=0$. In order to proceed to the proof of (8.9), we now show the following: For all odd $k\in\{3,\dots,n\}$ and $\ell\in\{0,\dots,k-2\}$ one has

$$\begin{aligned}&\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}C\right)^{k-2}R_{1+z}\right)\\&\quad=(-1)^{\ell}\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}\Phi\mathcal{Q}\right)^{\ell}\left(R_{1+z}C\right)^{k-2-\ell}R_{1+z}\right).\end{aligned}\tag{8.10}$$

In the beginning of the proof we have dealt with the case $k=3$. One notes that equation (8.10) always holds for $\ell=0$. Next, we assume that $k\in\{5,\dots,n\}$ is odd, such that equality (8.10) holds for some $\ell\in\{0,\dots,k-3\}$. Then one computes

$$\begin{aligned}&\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}C\right)^{k-2}R_{1+z}\right)\\&\quad=(-1)^{\ell}\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}\Phi\mathcal{Q}\right)^{\ell}\left(R_{1+z}C\right)^{k-2-\ell}R_{1+z}\right)\\&\quad=(-1)^{\ell}\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}\Phi\mathcal{Q}\right)^{\ell}R_{1+z}C\left(R_{1+z}C\right)^{k-2-\ell-1}R_{1+z}\right)\\&\quad=(-1)^{\ell}\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}\Phi\mathcal{Q}\right)^{\ell}R_{1+z}\left(\mathcal{Q}\Phi-\Phi\mathcal{Q}\right)\left(R_{1+z}C\right)^{k-2-(\ell+1)}R_{1+z}\right)\\&\quad=(-1)^{\ell}\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}\Phi\mathcal{Q}\right)^{\ell}\mathcal{Q}R_{1+z}\Phi\left(R_{1+z}C\right)^{k-2-(\ell+1)}R_{1+z}\right)\\&\qquad+(-1)^{\ell+1}\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}\Phi\mathcal{Q}\right)^{\ell+1}\left(R_{1+z}C\right)^{k-2-(\ell+1)}R_{1+z}\right).\end{aligned}$$

By Corollary A.9, the first term on the right-hand side cancels, proving equation (8.10). Putting $\ell=k-2$ in (8.10) implies the assertion. □

The following result is needed for Lemma 8.10, however, it is also of independent interest. Indeed, we will have occasion to use it rather frequently, when we discuss the case of three spatial dimensions specifically. Lemma 8.9 should be regarded as a regularization method, while preserving self-adjointness properties of the ($L^2$-realization) of the underlying operators:

**Lemma 8.9.** *Let $\varepsilon>0$, $n\in\mathbb{N}$, and $T\in\mathcal{B}\big(H^{-(n/2)-\varepsilon}(\mathbb{R}^n),H^{(n/2)+\varepsilon}(\mathbb{R}^n)\big)$. Recalling equation* (5.3)*, we consider*

$$t\colon\mathbb{R}^n\times\mathbb{R}^n\ni(x,y)\mapsto\left\langle\delta_{\{x\}},T\delta_{\{y\}}\right\rangle.$$

*For $\mu>0$ we denote $T_\mu:=(1-\mu\Delta)^{-1}T(1-\mu\Delta)^{-1}$ and $t_\mu$ correspondingly. Then, for all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$,*

$$t_\mu(x,y)\underset{\mu\downarrow0}{\longrightarrow}t(x,y).$$

*Proof.* It suffices to observe that for all $s\in\mathbb{R}$, $(1-\mu\Delta)^{-1}\underset{\mu\downarrow0}{\longrightarrow}I$ strongly in $H^s(\mathbb{R}^n)$ (see (5.1)). □

In order to proceed to prove the trace theorem, we need to investigate the asymptotic behavior of the integral kernel of $J_L^j(z)$ given by (7.6) on the diagonal. By Proposition 7.4 together with Lemma 5.6, we can use Gauss' divergence theorem for computing the integral over the diagonal (see also (5.5)). Thus, in the expression for the trace of $\chi_\Lambda B_L(z)$ we will use Gauss' theorem for the ball centered at 0 with radius $\Lambda$. Having applied the divergence theorem, we integrate over spheres of radius $\Lambda$. The volume element of the surface measure grows with $\Lambda^{n-1}$, so any term decaying faster than that will not contribute to the limit $\Lambda\to\infty$ in (7.4). Consequently, any estimate of integral kernels (or differences of such) to follow with the behavior of $|x|^{n-1+\gamma}$ for some $\gamma>0$ on the diagonal, can be neglected in the limit $\Lambda\to\infty$, when computing the expression $\lim_{\Lambda\to\infty}\operatorname{tr}(\chi_\Lambda B_L(z))$.

**Lemma 8.10.** *Let $n \in \mathbb{N}$ odd, $j \in \{1, \ldots, n\}$, $z \in \mathbb{C}$, $\mathrm{Re}(z) > -1$ and $R_{1+z}$ be given by* (4.6) *as well as $\mathcal{Q}$, $C$ and $\gamma_{j,n}$ given by* (6.3), (6.15) *and as in Remark* 6.1. *Then for $n \geqslant 3$, the integral kernel $h_{2,j}$ of*

$$2 \operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \Phi \left(R_{1+z} C\right)^{n-1} R_{1+z}\right)$$

*satisfies,*

$$h_{2,j}(x, x) = h_{3,j}(x, x) + g_{0,j}(x, x),$$

*where $h_{3,j}$ is the integral kernel of $2 \operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \Phi C^{n-1} R_{1+z}^n\right)$ and $g_{0,j}$ satisfies for some $\kappa > 0$,*

$$|g_{0,j}(x, x)| \leqslant \kappa(1 + |x|)^{1-n-\varepsilon}, \quad x \in \mathbb{R}^n,$$

*where $\varepsilon > 1/2$ is given as in Definition* 6.11. *In addition, if $n \geqslant 5$ and $z \in \mathbb{R}$, then the integral kernel $h_{1,j}$ of*

$$\operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \mathcal{Q} \left(R_{1+z} C\right)^{n-2} R_{1+z}\right)$$

*vanishes on the diagonal.*

*Proof.* We discuss $h_{1,j}$ first and consider the operator

$$B_n \coloneqq \left(\Phi \mathcal{Q} R_{1+z}\right)^{n-3} \Phi = \Phi \left(\mathcal{Q} R_{1+z} \Phi\right)^{n-3},$$

which is self-adjoint for all real $z > -1$. Indeed, this follows from the self-adjointness of $\Phi$ and the skew-self-adjointness of $\mathcal{Q} R_{1+z}$. For $\mu > 0$ define $B_{n,\mu} \coloneqq (1 - \mu\Delta)^{-1} B_n (1 - \mu\Delta)^{-1}$. Then the integral kernel $b_{n,\mu}$ of $B_{n,\mu}$ is continuous. Moreover, for all real $z > -1$, the opertor $B_{n,\mu}$ is self-adjoint, by the self-adjointness of $B_n$ and so $b_{n,\mu}$ is real and satisfies $b_{n,\mu}(x, y) = b_{n,\mu}(y, x)$ for all $x, y \in \mathbb{R}^n$. By Lemma 8.8 one recalls

$$\begin{aligned}
&\operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \mathcal{Q} \left(R_{1+z} C\right)^{n-2} R_{1+z}\right) \\
&\quad = -\operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \mathcal{Q} \left(R_{1+z} \Phi \mathcal{Q}\right)^{n-2} R_{1+z}\right) \\
&\quad = -\operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \mathcal{Q} R_{1+z} \left(\Phi \mathcal{Q} R_{1+z}\right)^{n-2}\right) \\
&\quad = -\operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \mathcal{Q} R_{1+z} \left(\Phi \mathcal{Q} R_{1+z}\right)^{n-3} \Phi \mathcal{Q} R_{1+z}\right) \\
&\quad = -\operatorname{tr}_{2^{\hat{n}}d} \left(\gamma_{j,n} \mathcal{Q} R_{1+z} B_n \mathcal{Q} R_{1+z}\right).
\end{aligned}$$

By Fubini's theorem and the symmetry of $B_{n,\mu}$, one has for all $j, k \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^n$, $z > -1$, $\mu > 0$,

$$\begin{aligned}
\Psi_{x,\mu}(j, k) &\coloneqq \int_{\mathbb{R}^n \times \mathbb{R}^n} (\partial_j r_{1+z})(x_1 - x) b_{n,\mu}(x_1, x_2) (\partial_k r_{1+z})(x_2 - x) \, d^n x_1 d^n x_2 \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} (\partial_j r_{1+z})(x_1 - x) b_{n,\mu}(x_2, x_1) (\partial_k r_{1+z})(x_2 - x) \, d^n x_1 d^n x_2 \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} (\partial_k r_{1+z})(x_1 - x) b_{n,\mu}(x_1, x_2) (\partial_j r_{1+z})(x_2 - x) \, d^n x_1 d^n x_2 \\
&= \Psi_{x,\mu}(k, j).
\end{aligned}$$

By Lemma 8.9 one has for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
h_{1,j}(x, y) = -\lim_{\mu \downarrow 0} \operatorname{tr}_{2^{\hat{n}}d} \Bigg( \sum_{i_2, i_3 = 1}^{n} \gamma_{j,n} \gamma_{i_2,n} \gamma_{i_3,n} \partial_{i_2} \int_{\mathbb{R}^n \times \mathbb{R}^n} & r_{1+z}(x - x_1) b_{n,\mu}(x_1, x_2) \\
& \times (\partial_{i_3} r_{1+z})(x_2 - y) \, d^n x_1 d^n x_2 \Bigg)
\end{aligned}$$

$$= \lim_{\mu\downarrow 0} \operatorname{tr}_{2^{\hat{n}}d}\bigg( \sum_{i_2,i_3=1}^{n} \gamma_{j,n}\gamma_{i_2,n}\gamma_{i_3,n} \int_{\mathbb{R}^n\times\mathbb{R}^n} (\partial_{i_2} r_{1+z})(x_1-x) b_{n,\mu}(x_1,x_2) \\ \times (\partial_{i_3} r_{1+z})(x_2-y)\, d^n x_1 d^n x_2 \bigg).$$

Thus, it follows from Corollary A.9 that

$$h_{1,j}(x,x) = \lim_{\mu\downarrow 0} \operatorname{tr}_{2^{\hat{n}}d}\Bigg( \sum_{i_2,i_3=1}^{n} \gamma_{j,n}\gamma_{i_2,n}\gamma_{i_3,n} \Psi_{x,\mu}(i_2,i_3)\Bigg) = 0, \quad x\in\mathbb{R}^n.$$

The assertion about $h_{2,j}$ is a direct consequence of Remark 5.18 and the asymptotic conditions imposed on $\Phi$. $\square$

For the estimate on the diagonal of the integral kernels of the operators under consideration in the next theorem we need to choose the real part of $z$ large. In fact, we use the Neumann series expression for the resolvents $(L^*L+z)^{-1}$ and $(LL^*+z)^{-1}$ and Remark 5.15, both of which making the choice of large $\operatorname{Re}(z)$ necessary. We shall also have an a priori bound on the argument of $z$, recalling the definition (8.8) of the sector $\Sigma_{z_0,\vartheta} \subset \mathbb{C}$.

**Theorem 8.11.** *Let $L = \mathcal{Q}+\Phi$ be given by* (7.1)*, and for $z\in\mathbb{C}_{\operatorname{Re}>-1}$, let $R_{1+z}$ be given by* (4.6) *and $C$ as in* (6.15)*. For $j\in\{1,\dots,n\}$, let $\gamma_{j,n}\in\mathbb{C}^{2^{\hat{n}}\times 2^{\hat{n}}}$ (cf. Remark* 6.1)*, and $\vartheta\in(0,\pi/2)$. Then there exists $z_0>0$, such that for all $z\in\Sigma_{z_0,\vartheta}$ (see* (8.8))*, the integral kernels $g_{1,j}$ and $g_{2,j}$ of the operators*

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\Phi\big( (L^*L+z)^{-1} + (LL^*+z)^{-1}\big) (CR_{1+z})^n\right)$$

*and*

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}Q\big( (L^*L+z)^{-1} + (LL^*+z)^{-1}\big) (CR_{1+z})^n\right),$$

*respectively, satisfy for some $\kappa>0$,*

$$\big[|g_{1,j}(x,x)| + |g_{2,j}(x,x)|\big] \leqslant \kappa(1+|x|)^{-n}, \quad x\in\mathbb{R}^n.$$

*Proof.* We choose $z_0$ such that $\sqrt{z_0} > 2n$ (one recalls Remark 5.15) and that for $M \coloneqq \sup_{x\in\mathbb{R}^n} \|\Phi(x)\| \vee \|(\mathcal{Q}\Phi)(x)\|$ one has $2M[z_0\cos(\vartheta)]^{-1/2} \leqslant 1/2$. We treat $g_{1,j}$ first. Let $z\in\Sigma_{z_0,\vartheta}$, then,

$$\begin{aligned} &\gamma_{j,n}\Phi\big( (L^*L+z)^{-1} + (LL^*+z)^{-1}\big) (CR_{1+z})^n \\ &\quad = \gamma_{j,n}\Phi 2\,(R_{1+z}C)^n \sum_{k=0}^{\infty} (R_{1+z}C)^{2k} R_{1+z} \\ &\quad = \sum_{k=0}^{\infty} \gamma_{j,n}\Phi 2\,(R_{1+z}C)^n (R_{1+z}C)^{2k} R_{1+z}. \end{aligned}$$

For $x\in\mathbb{R}^n$ one infers (recalling $\delta_{\{x\}}$ in (5.3)),

$$\begin{aligned} g_{1,j}(x,x) &= \left\langle \delta_{\{x\}}, \sum_{k=0}^{\infty} \gamma_{j,n}\Phi 2\,(R_{1+z}C)^{2k} (R_{1+z}C)^n R_{1+z}\delta_{\{x\}} \right\rangle \\ &= \sum_{k=0}^{\infty} \big\langle \delta_{\{x\}}, \gamma_{j,n}\Phi 2\,(R_{1+z}C)^{2k} (R_{1+z}C)^n R_{1+z}\delta_{\{x\}}\big\rangle. \end{aligned}$$

Hence, by Lemma 5.14 together with Remark 5.15, there exists $c>0$ such that for all $x \in \mathbb{R}^n$,

$$\left|\left\langle \delta_{\{x\}}, \gamma_{j,n} \Phi 2 \left(R_{1+z}C\right)^{2k} \left(R_{1+z}C\right)^n R_{1+z}\delta_{\{x\}}\right\rangle\right| \leqslant c\left(\frac{2M}{\sqrt{1+z_0}}\right)^{2k}\left(\frac{1}{1+|x|}\right)^n.$$

Since $2M[1+z_0]^{-1/2} \leqslant 1/2$, one concludes that

$$\begin{aligned}|g_{1,j}(x,x)| &\leqslant \sum_{k=0}^{\infty}\left|\left\langle \delta_{\{x\}}, \gamma_{j,n} \Phi 2 \left(R_{1+z}C\right)^{2k} \left(R_{1+z}C\right)^n R_{1+z}\delta_{\{x\}}\right\rangle\right| \\ &\leqslant \sum_{k=0}^{\infty} c\left(\frac{2M}{\sqrt{1+z_0}}\right)^{2k}\left(\frac{1}{1+|x|}\right)^n \leqslant c\left(\frac{1}{1+|x|}\right)^n.\end{aligned}$$

The analogous reasoning applies to $g_{2,j}$. $\square$

We conclude the results on estimates of certain integral kernels on the diagonal with the following corollary, which, roughly speaking, says that the diagonal of the integral kernels involved is determined by the integral kernel of the operator to be discussed in Proposition 8.13.

**Corollary 8.12.** *For $z \in \mathbb{C}$, $\mathrm{Re}(z) > -1$, denote $R_{1+z}$ as in* (4.6)*, let $\Phi\colon \mathbb{R}^n \to \mathbb{C}^{d\times d}$ be admissible* (*see Definition* 6.11)*, and $L = \mathcal{Q} + \Phi$ as in* (7.1)*, $\vartheta \in (0,\pi/2)$. In addition, denote $J_L^j(z)$ for $z \in \varrho(-L^*L)\cap\varrho(-LL^*)$ as in* (7.6) *for all $j \in \{1,\dots,n\}$, and $C$ as in* (6.15)*. Moreover, let $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}$, $j \in \{1,\dots,n\}$* (*cf. Remark* 6.1).

(*i*) *Let $n \in \mathbb{N}_{\geqslant 5}$, $j \in \{1,\dots,n\}$. Then there exists $z_0 > 0$, such that if $z \in \Sigma_{z_0,\vartheta}$* (*see* (8.8))*, and $h$ and $g$ denote the integral kernel of $2\,\mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\Phi C^{n-1}R_{1+z}^n\right)$ and $J_L^j(z)$, respectively, then for some $\kappa > 0$,*

$$|h(x,x) - g(x,x)| \leqslant \kappa(1+|x|)^{1-n-\varepsilon}, \quad x \in \mathbb{R}^n,$$

*where $\varepsilon > 1/2$ is given as in Definition* 6.11.

(*ii*) *The assertion of part* (*i*) *also holds for $n = 3$, if, in the above statement, $J_L^j(z)$ is replaced by $J_L^j(z) - 2\,\mathrm{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}\right)$.*

*Proof.* One recalls from Lemma 7.7,

$$\begin{aligned}J_L^j(z) &= 2\,\mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}C\right)^{n-2}R_{1+z}\right) + 2\,\mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\Phi\left(R_{1+z}C\right)^{n-1}R_{1+z}\right) \\ &\quad + \mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\big(\left(L^*L+z\right)^{-1} + \left(LL^*+z\right)^{-1}\big)\left(CR_{1+z}\right)^n\right) \\ &\quad + \mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\Phi\big(\left(L^*L+z\right)^{-1} + \left(LL^*+z\right)^{-1}\big)\left(CR_{1+z}\right)^n\right).\end{aligned}$$

With the help of Theorem 8.11 one deduces that the integral kernels of the last two terms may be estimated by $\kappa(1+|x|)^{-n}$ on the diagonal. The integral kernel of the first term on the right-hand side vanishes on the diagonal, which is asserted in Lemma 8.10. Hence, it remains to inspect the second term of the right-hand side. Thus, the assertion follows from Lemma 8.10. $\square$

Having identified the integral kernel $g_j$ of $2\,\mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\Phi C^{n-1}R_{1+z}^n\right)$ to be the only term determining the trace of $\chi_\Lambda B_L(z)$ as $\Lambda \to \infty$, we shall compute the integral over the diagonal of $g_j$:

**Proposition 8.13.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $C$ as in* (6.15)*, $z \in \mathbb{C}$, $\mathrm{Re}(z) > -1$, with $R_{1+z}$ given by* (4.6)*, $\Phi \colon \mathbb{R}^n \to \mathbb{C}^{d\times d}$ be admissible (see Definition* 6.11*), $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}$, $j \in \{1,\dots,n\}$, as in Remark* 6.1*. Then for $j \in \{1,\dots,n\}$, the integral kernel $g_j$ of*

$$2\,\mathrm{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\Phi C^{n-1}R_{1+z}^n\right)$$

*satisfies,*

$$\begin{aligned} g_j(x,x) = {} & (1+z)^{-n/2}\left(\frac{i}{8\pi}\right)^{(n-1)/2}\frac{1}{[(n-1)/2]!} \\ & \times \sum_{i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}}\,\mathrm{tr}\left(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)\right), \quad x\in\mathbb{R}^n, \end{aligned}$$

*where $\varepsilon_{ji_1\dots i_{n-1}}$ denotes the $\varepsilon$-symbol as in Proposition* A.8.

*Proof.* We recall that $n = 2\widehat{n}+1$. With the help of Proposition A.8, $g_j$ is given by

$$\begin{aligned} (x,y) \mapsto {} & 2(2i)^{\widehat{n}} \sum_{i_1\dots i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}}\,\mathrm{tr}\left(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)\right) \\ & \times \int_{(\mathbb{R}^n)^{n-1}} r_{1+z}(x-x_1)r_{1+z}(x_1-x_2)\cdots r_{1+z}(x_{n-1}-y)\,d^nx_1\cdots d^nx_{n-1}. \end{aligned}$$

Hence, by substitution in the integral expression and putting $x = y$, one obtains

$$\begin{aligned} g_j(x,x) = {} & 2(2i)^{\widehat{n}} \sum_{i_1\dots i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}}\,\mathrm{tr}\left(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)\right) \\ & \times \int_{(\mathbb{R}^n)^{n-1}} r_{1+z}(x_1)r_{1+z}(x_1-x_2)\cdots r_{1+z}(x_{n-1})\,d^nx_1\cdots d^nx_{n-1}. \end{aligned}$$

The last integral can be computed with the help of the Fourier transform and polar coordinates, as was done in Proposition 5.8. In fact, one gets (see also [57, 3.252.2]),

$$\begin{aligned} & \int_{(\mathbb{R}^n)^{n-1}} r_{1+z}(x_1)r_{1+z}(x_1-x_2)\cdots r_{1+z}(x_{n-1})\,d^nx_1\cdots d^nx_{n-1} \\ & \quad = (2\pi)^{-n}\frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty r^{n-1}\frac{1}{(r^2+1+z)^n}\,dr \\ & \quad = (2\pi)^{-n}\frac{2\pi^{n/2}}{\Gamma(n/2)}(1+z)^{-n/2}\frac{2^{-n}\sqrt{\pi}\Gamma(n/2)}{[(n-1)/2]!} \\ & \quad = \frac{1}{2^{2n-1}}\frac{1}{\pi^{(n-1)/2}}\frac{1}{[(n-1)/2]!}(1+z)^{-n/2}, \end{aligned}$$

and notes that

$$\begin{aligned} & 2(2i)^{(n-1)/2}\frac{1}{2^{2n-1}}\frac{1}{\pi^{(n-1)/2}}\frac{1}{[(n-1)/2]!} \\ & \quad = \left(\frac{i}{\pi}\right)^{(n-1)/2}\frac{1}{2^{(4n-4-n+1)/2}}\frac{1}{[(n-1)/2]!} \\ & \quad = \left(\frac{i}{\pi}\right)^{(n-1)/2}\frac{1}{2^{(3n-3)/2}}\frac{1}{[(n-1)/2]!} \\ & \quad = \left(\frac{i}{8\pi}\right)^{(n-1)/2}\frac{1}{[(n-1)/2]!}. \end{aligned}$$

$\square$

Finally, we are ready to prove the (trace) Theorem 7.1, for $n \geqslant 5$, that is, we consider the operator $L = \mathcal{Q} + \Phi$ with an admissible potential $\Phi$, such that $\Phi$ is smooth and attains values in the self-adjoint, unitary $d\times d$-matrices. In addition, we recall that the first derivatives of $\Phi$ behave like $|x|^{-1}$ for large $x$, whereas higher-order derivatives decay at least with the behavior $|x|^{-1-\varepsilon}$ for large $x$ and some $\varepsilon > 1/2$. We note that we already established the Fredholm property of $L$ in Theorem 6.3. We outline the proof of Theorem 7.1, for $n \geqslant 5$, as follows. The results in Section 7 yield the applicability of Theorem 3.4. More precisely, the operator $\chi_\Lambda B_L(z)$ is trace class with trace computable as the integral over the diagonal of the integral kernel of $\chi_\Lambda B_L(z)$. With Proposition 7.4 we will deduce that only the term involving $J_L^j(z)$, being analysed in Lemma 7.7, matters for the computation of the index. Next, we will show that $\{z \mapsto \operatorname{tr}(\chi_\Lambda B_L(z))\}_{\Lambda>0}$ is locally bounded using Lemma 5.6 (in particular (5.5)). The local boundedness result is then obtained via Gauss' divergence theorem and Lemma 8.10 as well as Theorem 8.7. Having proved local boundedness, we will use Montel's theorem for deducing that at least for a sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ the limit $f \coloneqq \lim_{k\to\infty} \operatorname{tr}(\chi_{\Lambda_k} B_L(\cdot))$ exists in the compact open topology, that is, the topology of uniform convergence on compacts. With the results from Corollary 8.12 and Proposition 8.13, choosing $\operatorname{Re}(z)$ sufficiently large, we get an explicit expression for $f$. The explicit expression for $f$, by the principle of analytic continuation, carries over to $z$ in a neighborhood of 0. As we know, by Theorem 3.4, that the limit $\lim_{\Lambda\to\infty} \operatorname{tr}(\chi_\Lambda B_L(0))$ exists and coincides with the index of $L$, we can then deduce that not only for the sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ but, in fact, the limit $\lim_{\Lambda\to\infty} \operatorname{tr}(\chi_\Lambda B_L(\cdot))$ exists in the compact open topology and coincides with $f$ given in (7.4). The detailed arguments read as follows.

*Proof of Theorem* 7.1 *for* $n \geqslant 5$. By Theorem 7.8, $\chi_\Lambda B_L(z)$ is trace class for every $\Lambda > 0$. Moreover, by Remark 7.9, $\operatorname{tr}(\chi_\Lambda B_L(z))$ can be computed as the integral over the diagonal of the respective integral kernel. Hence, by Proposition 7.4, equation (7.5), recalling also Remark 5.2, one obtains

$$\begin{aligned}
2\operatorname{tr}(\chi_\Lambda B_L(z)) &= 2\int_{B(0,\Lambda)} \big\langle \delta_{\{x\}}, B_L(z)\delta_{\{x\}}\big\rangle_{H^{-(n/2)-\varepsilon},H^{(n/2)+\varepsilon}}\; d^n x \\
&= \int_{B(0,\Lambda)} \Big\langle \delta_{\{x\}}, \Big(\sum_{j=1}^n \big[\partial_j, J_L^j(z)\big] + A_L(z)\Big)\delta_{\{x\}}\Big\rangle\, d^n x \\
&= \int_{B(0,\Lambda)} \Big\langle \delta_{\{x\}}, \sum_{j=1}^n \big[\partial_j, J_L^j(z)\big]\delta_{\{x\}}\Big\rangle\, d^n x, \qquad (8.11)
\end{aligned}$$

where we used Lemma 8.1 to deduce that $\big\langle \delta_{\{x\}}, A_L(z)\delta_{\{x\}}\big\rangle = 0$ for all $x \in \mathbb{R}^n$. Next, we prove that $\{z \mapsto \operatorname{tr}(\chi_\Lambda B_L(z))\}_{\Lambda>0}$ is locally bounded. One recalls from Lemma 7.7,

$$\begin{aligned}
J_L^j(z) &= 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}\,(R_{1+z}C)^{n-2}\,R_{1+z}\big) + 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi\,(R_{1+z}C)^{n-1}\,R_{1+z}\big) \\
&\quad + \operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}\big(\,(L^*L+z)^{-1} + (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big) \\
&\quad + \operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi\big(\,(L^*L+z)^{-1} + (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big).
\end{aligned}$$

Hence,

$$\sum_{j=1}^n \big[\partial_j, J_L^j(z)\big] = \sum_{j=1}^n \big[\partial_j, \big(2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}(R_{1+z}C)^{n-2}R_{1+z}\big)$$

$$
\begin{aligned}
&\quad + 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi(R_{1+z}C)^{n-1}R_{1+z}\big)\big)\big] \\
&+ \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q},\mathcal{Q}\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n}\,\big]\big) \\
&+ \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q},\Phi\big(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\big)\,(CR_{1+z})^{n}\,\big]\big).
\end{aligned}
\tag{8.12}
$$

Denoting by $h_j$ the integral kernel of $2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi C^{n-1}R_{1+z}^n\big)$, one observes that for some constant $\kappa > 0$, $|h_j(x,x)| \leqslant \kappa(1+|x|)^{1-n}$, $x \in \mathbb{R}^n$. Hence, for any $\Lambda > 0$, invoking Lemma 5.6, Gauss' theorem implies that

$$
\begin{aligned}
&\bigg|\int_{B(0,\Lambda)}\bigg\langle \delta_{\{x\}},\sum_{j=1}^{n}\big[\partial_j, 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi C^{n-1}R_{1+z}^n\big)\big]\,\delta_{\{x\}}\bigg\rangle\, d^n x\bigg| \\
&\quad = \bigg|\int_{B(0,\Lambda)}\sum_{j=1}^{n}(\partial_j h_j)(x,x)\, d^n x\bigg| \\
&\quad = \bigg|\int_{\Lambda S^{n-1}}\sum_{j=1}^{n} h_j(x,x)\frac{x_j}{\Lambda}\, d^{n-1}\sigma(x)\bigg| \\
&\quad \leqslant \int_{\Lambda S^{n-1}}\sum_{j=1}^{n} |h_j(x,x)|\, d^{n-1}\sigma(x) \\
&\quad \leqslant n\kappa(1+\Lambda)^{1-n}\Lambda^{n-1}\omega_{n-1},
\end{aligned}
\tag{8.13}
$$

(with $\omega_{n-1}$ being the $(n-1)$-dimensional volume of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$, see (5.6)). The latter is uniformly bounded with respect to $\Lambda > 0$.

Using Lemma 8.10, the definition of $g_{0,j}$ in that lemma as well as Gauss' theorem, one arrives at

$$
\begin{aligned}
&\bigg|\int_{B(0,\Lambda)}\bigg\langle \delta_{\{x\}},\sum_{j=1}^{n}\big[\partial_j, 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}\,(R_{1+z}C)^{n-2}\,R_{1+z}\big)\big]\delta_{\{x\}}\bigg\rangle\, d^n x \\
&\qquad + \int_{B(0,\Lambda)}\bigg\langle \delta_{\{x\}},\sum_{j=1}^{n}\big[\partial_j,\big(2\operatorname{tr}_{2^{\hat n}d}\gamma_{j,n}\Phi\,(R_{1+z}C)^{n-1}\,R_{1+z}\big)\big]\delta_{\{x\}}\bigg\rangle\, d^n x \\
&\qquad - \int_{B(0,\Lambda)}\bigg\langle \delta_{\{x\}},\sum_{j=1}^{n}\big[\partial_j, 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi C^{n-1}R_{1+z}^n\big)\,\big]\delta_{\{x\}}\bigg\rangle\, d^n x\bigg| \\
&\quad = \bigg|\int_{B(0,\Lambda)}\sum_{j=1}^{n}(\partial_j g_{0,j})(x,x)\, d^n x\bigg| \\
&\quad \leqslant \bigg|\int_{\Lambda S^{n-1}}\sum_{j=1}^{n} g_{0,j}(x,x)\frac{x_j}{\Lambda} d^{n-1}\sigma(x)\bigg| \\
&\quad \leqslant \int_{\Lambda S^{n-1}}\sum_{j=1}^{n} |g_{0,j}(x,x)|\, d^{n-1}\sigma(x) \\
&\quad \leqslant \int_{\Lambda S^{n-1}} n\kappa(1+|x|)^{1-n-\varepsilon} d^{n-1}\sigma(x) \\
&\quad \leqslant n\kappa(1+\Lambda)^{1-n-\varepsilon}\omega_{n-1}\Lambda^{n-1} \underset{\Lambda\to\infty}{\longrightarrow} 0.
\end{aligned}
\tag{8.14}
$$

Next, Theorem 8.7 implies that

$$\Big\{z \mapsto z \operatorname{tr}\Big(\chi_\Lambda\Big( \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q}, \mathcal{Q}\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big]\big) \\ + \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q}, \Phi\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big]\big)\Big)\Big)\Big\}_{\Lambda>0} \tag{8.15}$$

is bounded on any compact neighborhood of 0 intersected with $B(0,\delta)\cup(\varrho(-LL^*)\cap \varrho(-L^*L))$ for some $\delta > 0$. Hence, summarizing equations (8.11) and (8.12), we get for $z \in \mathbb{C}_{\mathrm{Re}>-1} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$:

$$\begin{aligned} z2\operatorname{tr}(\chi_\Lambda B_L(z)) &= z\int_{B(0,\Lambda)} \Big\langle \delta_{\{x\}}, \sum_{j=1}^n \big[\partial_j, J_L^j(z)\big]\delta_{\{x\}}\Big\rangle\, d^n x \\ &= z\int_{B(0,\Lambda)} \Big\langle \delta_{\{x\}}, \sum_{j=1}^n \big[\partial_j, 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}(R_{1+z}C)^{n-2}R_{1+z}\big)\big]\delta_{\{x\}}\Big\rangle\, d^n x, \\ &\quad + z\int_{B(0,\Lambda)} \Big\langle \delta_{\{x\}}, \sum_{j=1}^n \big[\partial_j, 2\operatorname{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\Phi(R_{1+z}C)^{n-1}R_{1+z}\big)\big]\delta_{\{x\}}\Big\rangle\, d^n x \\ &\quad + z\int_{B(0,\Lambda)}\Big\langle \delta_{\{x\}}, \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q}, \mathcal{Q}\big(\,(L^*L+z)^{-1} \\ &\qquad\qquad + (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big]\big)\delta_{\{x\}}\Big\rangle\, d^n x \\ &\quad + z\int_{B(0,\Lambda)}\Big\langle \delta_{\{x\}}, \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q}, \Phi\big(\,(L^*L+z)^{-1} \\ &\qquad\qquad + (LL^*+z)^{-1}\,\big)\,(CR_{1+z})^n\,\big]\big)\delta_{\{x\}}\Big\rangle\, d^n x \\ &= z\int_{B(0,\Lambda)} \sum_{j=1}^n (\partial_j g_{0,j})(x,x)\, d^n x + z\int_{B(0,\Lambda)} \sum_{j=1}^n (\partial_j h_j)(x,x)\, d^n x + \\ &\quad + z\operatorname{tr}\Big(\chi_\Lambda\Big( \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q}, \mathcal{Q}\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big]\big) \\ &\qquad\qquad + \operatorname{tr}_{2^{\hat n}d}\big(\big[\mathcal{Q}, \Phi\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{1+z})^n\big]\big)\Big)\Big). \end{aligned}$$

Thus, with the estimates (8.13) and (8.14) together with (8.15), one infers that

$$\{z \mapsto 2z\operatorname{tr}(\chi_\Lambda B_L(z))\}_{\Lambda>0}$$

is locally bounded on $B(0,\delta)\cup\mathbb{C}_{\mathrm{Re}>0}$ for some $\delta > 0$. By Lemma 8.3 together with Theorem 7.8, one infers that

$$\{z \mapsto 2\operatorname{tr}(\chi_\Lambda B_L(z))\}_{\Lambda>0}$$

is locally bounded on $B(0,\delta)\cup\mathbb{C}_{\mathrm{Re}>0}$. By Montel's Theorem, there exists a sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ of positive reals tending to infinity such that

$$\{z \mapsto 2\operatorname{tr}(\chi_{\Lambda_k} B_L(z))\}_{k\in\mathbb{N}}$$

converges in the compact open topology. We denote by $f$ the respective limit. Then Lemma 5.6 implies that for $k \in \mathbb{N}$,

$$2\operatorname{tr}(\chi_{\Lambda_k} B_L(z)) = \int_{B(0,\Lambda_k)} \sum_{j=1}^n (\partial_j g_j)(x,x)\, d^n x.$$

and so

$$f(z) = \lim_{k\to\infty} \int_{B(0,\Lambda_k)} \operatorname{div} \mathbb{G}_{J,z}(x)\, d^n x.$$

Here we denote $\mathbb{G}_{J,z} \coloneqq \{x \mapsto g_j(x,x)\}_{j\in\{1,\dots,n\}}$, with $g_j$ being the integral kernel of $J_L^j(z)$ for $j \in \{1,\dots,n\}$. Next, let $\vartheta \in (0,\pi/2)$ and choose $z_0 > 0$ as in Corollary 8.12 $(i)$. Let $z \in \Sigma_{z_0,\vartheta}$, see (8.8). Recalling that $h_j$ is the integral kernel of $2\mathrm{tr}_{2^{\hat{n}}d}\big(\gamma_{j,n}\Phi C^{n-1} R_{1+z}^n\big)$, we define $\mathbb{H}_z \coloneqq \{x \mapsto h_j(x,x)\}_{j\in\{1,\dots,n\}}$. Due to Corollary 8.12, one can find $\kappa > 0$ such that for $k \in \mathbb{N}$,

$$\begin{aligned}
&\left| \int_{\Lambda_k S^{n-1}} \left( (\mathbb{G}_{J,z} - \mathbb{H}_z)(x), \frac{x}{\Lambda_k} \right)_{\mathbb{R}^n} d^{n-1}\sigma(x) \right| \\
&\quad \leqslant \int_{\Lambda_k S^{n-1}} \|(\mathbb{G}_{J,z} - \mathbb{H}_z)(x)\|_{\mathbb{R}^n}\, d^{n-1}\sigma(x) \\
&\quad \leqslant \kappa \int_{\Lambda_k S^{n-1}} (1+|x|)^{1-n-\varepsilon}\, d^{n-1}\sigma(x) \\
&\quad = \kappa \Lambda_k^{n-1} \omega_{n-1} (1+\Lambda_k)^{1-n-\varepsilon}.
\end{aligned}$$

Consequently,

$$\lim_{k\to\infty} \int_{\Lambda_k S^{n-1}} \left( (\mathbb{G}_{J,z} - \mathbb{H}_z)(x), \frac{x}{\Lambda_k} \right)_{\mathbb{R}^n} d^{n-1}\sigma(x) = 0.$$

Hence, with the help of Gauss' theorem,

$$\begin{aligned}
f(z) &= \lim_{k\to\infty} \int_{B(0,\Lambda_k)} \sum_{j=1}^n (\partial_j g_j)(x,x)\, d^n x = \int_{\mathbb{R}^n} \operatorname{div} \mathbb{G}_{J,z}(x)\ d^n x \\
&= \lim_{k\to\infty} \int_{B(0,\Lambda_k)} \operatorname{div} \mathbb{G}_{J,z}(x)\, d^n x = \lim_{k\to\infty} \int_{\Lambda_k S^{n-1}} \left( \mathbb{G}_{J,z}(x), \frac{x}{\Lambda_k} \right)_{\mathbb{R}^n} d^{n-1}\sigma(x) \\
&= \lim_{k\to\infty} \int_{\Lambda_k S^{n-1}} \left( \mathbb{H}_z(x), \frac{x}{\Lambda_k} \right)_{\mathbb{R}^n} d^{n-1}\sigma(x) \\
&= \left(\frac{i}{8\pi}\right)^{(n-1)/2} \frac{1}{[(n-1)/2]!} (1+z)^{-n/2} \lim_{k\to\infty} \int_{\Lambda_k S^{n-1}} \\
&\quad \times \sum_{j=1}^n \left( \sum_{i_1,\dots,i_{n-1}=1}^n \varepsilon_{j i_1 \dots i_{n-1}} \operatorname{tr}\big(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)\big) \right) \\
&\qquad \times \left( \frac{x_j}{\Lambda_k} \right) d^{n-1}\sigma(x),
\end{aligned} \tag{8.16}$$

where, for the last integral, we used Proposition 8.13. By Theorem 3.4 one has $f(0) = 2\,\mathrm{ind}(L)$. In particular, any sequence $\{\Lambda_k\}_k$ of positive reals converging to infinity contains a subsequence $\{\Lambda_{k_\ell}\}_\ell$ such that for that particular subsequence the limit

$$\begin{aligned}
&\lim_{\ell\to\infty} \int_{\Lambda_{k_\ell} S^{n-1}} \sum_{j=1}^n \left( \sum_{i_1,\dots,i_{n-1}=1}^n \varepsilon_{j i_1 \dots i_{n-1}} \operatorname{tr}\big(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)\big) \right) \\
&\qquad \times \left( \frac{x_j}{\Lambda_{k_\ell}} \right) d^{n-1}\sigma(x)
\end{aligned}$$

exists and equals

$$\frac{2\,\mathrm{ind}(L)\,[(n-1)/2]!}{[i/(8\pi)]^{(n-1)/2}}. \tag{8.17}$$

Hence, the limit

$$\begin{aligned}\lim_{\Lambda\to\infty}\int_{\Lambda S^{n-1}}\sum_{j=1}^{n}\Bigg(&\sum_{i_1,\dots,i_{n-1}=1}^{n}\varepsilon_{ji_1\dots i_{n-1}}\,\mathrm{tr}\left(\Phi(x)\left(\partial_{i_1}\Phi\right)(x)\dots\left(\partial_{i_{n-1}}\Phi\right)(x)\right)\Bigg)\\&\times\left(\frac{x_j}{\Lambda}\right)d^{n-1}\sigma(x)\end{aligned} \tag{8.18}$$

exists and equals the number in (8.17). On the other hand, for $z\in\Sigma_{z_0,\vartheta}$, (see again Corollary 8.12) the family

$$\{z\mapsto \mathrm{tr}(\chi_\Lambda B_L(z))\}_{\Lambda>0}$$

converges for $\Lambda\to\infty$ on the domain $\Sigma_{z_0,\vartheta}$ if and only if the limit in (8.18) exists. Indeed, this follows from the explicit expression for the limit in (8.16). Therefore,

$$\{z\mapsto \mathrm{tr}(\chi_\Lambda B_L(z))\}_{\Lambda>0}$$

converges in the compact open topology on $\Sigma_{z_0,\vartheta}$. By the local boundedness of the latter family on the domain $B(0,\delta)\cup\mathbb{C}_{\mathrm{Re}>0}$, the principle of analytic continuation implies that the latter family actually converges on the domain $B(0,\delta)\cup\mathbb{C}_{\mathrm{Re}>0}$ in the compact open topology. In particular,

$$\begin{aligned}&2f(z)(1+z)^{n/2}\frac{[(n-1)/2]!}{[i/(8\pi)]^{(n-1)/2}}\\&\quad=\lim_{\Lambda\to\infty}\int_{\Lambda S^{n-1}}\sum_{j=1}^{n}\Bigg(\sum_{i_1,\dots,i_{n-1}=1}^{n}\varepsilon_{ji_1\dots i_{n-1}}\,\mathrm{tr}\left(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)\right)\Bigg)\\&\qquad\qquad\times\left(\frac{x_j}{\Lambda}\right)d^{n-1}\sigma(x).\end{aligned}$$

$\square$

## 9. The Case $n = 3$

In this section we shall discuss the necessary modifications, such that Theorem 7.1 continues to hold also for the case $n = 3$. The main issue for the need of extra arguments for this case is the lack of differentiability of the integral kernel of $J_L^j(z)$ given in Lemma 8.1. The main issue being the first summand in the expression for $J_L^j(z)$ derived in Lemma 7.7, that is, the term

$$\operatorname{tr}_{2^{\widehat{n}}d}\big(\gamma_{j,n}\mathcal{Q}(R_{1+z}C)R_{1+z}\big),$$

for the integral kernel of which we fail to show differentiability. Indeed, as this opertor increases regularity only by 3 orders of differentiability, not even continuity of the associated integral kernel is clear. The basic idea to overcome this difficulty and to get the result asserted in Theorem 7.1 also for the case $n = 3$ has already been used and is contained in Lemma 8.9. So, the operator $B_L(z)$ will be multiplied from the left and from the right by $(1 - \mu\Delta)^{-1}$ for some $\mu > 0$. The reason for multiplying from both sides is that we wanted to re-use strategies for showing that certain integral kernels vanish on the diagonal. The key for the latter arguments has been the self-adjointness of the operators under consideration, which, in turn, result in symmetry properties for the associated integral kernel.

An additional fact, enabling the strategy just sketched for the case $n = 3$, is the following result.

**Proposition 9.1** (See, e.g., [92], p. 28–29, or [105], Lemma 6.1.3)**.** *Assume $\mathcal{H}$ is a complex, separable Hilbert space, $B \in \mathcal{B}_1(\mathcal{H})$, and $A \geqslant 0$ is self-adjoint in $\mathcal{H}$. Then for $\mu > 0$, $B_\mu := (1+\mu A)^{-1}B(1+\mu A)^{-1} \in \mathcal{B}_1(\mathcal{H})$ and $B_\mu \to B$ in $\mathcal{B}_1(\mathcal{H})$ as $\mu \downarrow 0$. In particular, $\operatorname{tr}_{\mathcal{H}}(B_\mu) \underset{\mu\downarrow 0}{\longrightarrow} \operatorname{tr}_{\mathcal{H}}(B)$.*

Next, we will give the details for the modifications of the proof of Theorem 7.1 for the case $n = 3$. Thus, for $\mu > 0$, we introduce the operator

$$B_{L,\mu}(z) := (1-\mu\Delta)^{-1}\, B_L(z)\,(1-\mu\Delta)^{-1}, \tag{9.1}$$

where $z \in \varrho\,(-L^*L) \cap \varrho\,(-LL^*)$, $L = \mathcal{Q} + \Phi$ given by (7.1), and $B_L(z)$ is given by (7.2). We also introduce

$$\begin{aligned} J^j_{L,\mu}(z) = (1-\mu\Delta)^{-1}\Big(&\operatorname{tr}_{2^{\widehat{n}}d}\big(L\big(L^*L+z\big)^{-1}\gamma_{j,n}\big) \\ &+ \operatorname{tr}_{2^{\widehat{n}}d}\big(L^*(LL^*+z)^{-1}\gamma_{j,n}\big)\Big)(1-\mu\Delta)^{-1}, \end{aligned} \tag{9.2}$$

with $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}$, $j \in \{1,\dots,n\}$, as in Remark 6.1, and

$$\begin{aligned} A_{L,\mu}(z) = (1-\mu\Delta)^{-1}\Big(&\operatorname{tr}_{2^{\widehat{n}}d}\big(\big[\Phi, L^*(LL^*+z)^{-1}\big]\big) \\ &- \operatorname{tr}_{2^{\widehat{n}}d}\big(\big[\Phi, L(L^*L+z)^{-1}\big]\big)\Big)(1-\mu\Delta)^{-1} \end{aligned} \tag{9.3}$$

for the admissible potential $\Phi$ (see Definition 6.11). By Theorem 7.8 and the ideal property of $\mathcal{B}_1(\mathcal{H})$, the operator $B_{L,\mu}(z)$ is trace class for all $\mu > 0$ and $z \in \varrho\,(-L^*L) \cap \varrho\,(-LL^*)$ and $\operatorname{Re}(z) > -1$. As for the case $n \geqslant 5$, we need the following more detailed description of the operator $B_{L,\mu}(z)$:

**Lemma 9.2.** *Let $L = \mathcal{Q} + \Phi$ as in* (7.1)*, $\mu > 0$, $z \in \varrho\,(-L^*L) \cap \varrho\,(-LL^*)$, with $\operatorname{Re}(z) > -1$. Then with $J^j_{L,\mu}(z)$, $j \in \{1,\dots,m\}$, and $A_{L,\mu}(z)$ given by* (9.2) *and* (9.3)*, one has*

$$2B_{L,\mu}(z) = \sum_{j=1}^{n}\big[\partial_j, J^j_{L,\mu}(z)\big] + A_{L,\mu}(z).$$

*Proof.* The only nontrivial item to be established, invoking Proposition 7.4 together with equations (7.5), (7.6), and (7.7), is to establish that for $j \in \{1, \dots, n\}$,

$$\left[\partial_j, J^j_{L,\mu}(z)\right] = (1-\mu\Delta)^{-1}\left[\partial_j, J^j_L(z)\right](1-\mu\Delta)^{-1}.$$

Recalling $\gamma_{j,n} \in \mathbb{C}^{2^{\hat{n}}\times 2^{\hat{n}}}$, $j \in \{1, \dots, n\}$ (cf. Remark 6.1), one observes that

$$\begin{aligned}
&(1-\mu\Delta)^{-1}\left[\partial_j, \operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right)\right](1-\mu\Delta)^{-1} \\
&\quad = (1-\mu\Delta)^{-1}\partial_j \operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right) \\
&\qquad - \operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right)\partial_j(1-\mu\Delta)^{-1} \\
&\quad = \partial_j(1-\mu\Delta)^{-1}\operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right)(1-\mu\Delta)^{-1} \\
&\qquad - (1-\mu\Delta)^{-1}\operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right)(1-\mu\Delta)^{-1}\partial_j \\
&\quad = \left[\partial_j, (1-\mu\Delta)^{-1}\operatorname{tr}_{2^{\hat{n}}d}\left(L\left(L^*L+z\right)^{-1}\gamma_{j,n}\right)(1-\mu\Delta)^{-1}\right],
\end{aligned}$$

yielding the assertion. □

In contrast to the operator $J^j_L(z)$, the integral kernel for the regularized operator $J^j_{L,\mu}(z)$ satisfies the desired differentiability properties:

**Corollary 9.3.** *Let $L = \mathcal{Q} + \Phi$ be given by* (7.1)*, $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$, with $\operatorname{Re}(z) > -1$, and suppose $\mu > 0$. If $n \in \mathbb{N}$ is odd, then for all $j \in \{1, \dots, n\}$, the integral kernel of $J^j_{L,\mu}(z)$ given by* (9.2) *is continuously differentiable.*

*Proof.* We recall $R_{1+z}$ as given by (4.6), $\mathcal{Q}$ and $C$ given by (6.3) and (6.15), respectively, as well as $\gamma_{j,n}$ as in Remark 6.1. According to Proposition 5.4 for $\ell \in \mathbb{R}$, it suffices to observe that the operator

$$(1-\mu\Delta)^{-1}\operatorname{tr}_{2^{\hat{n}}d}\gamma_{j,n}\mathcal{Q}\left(R_{1+z}C\right)^{n-2}R_{1+z}(1-\mu\Delta)^{-1}$$

is continuous from $H^\ell(\mathbb{R}^n)$ (see (5.1)) to $H^{\ell+2(n-2)+2+4-1}(\mathbb{R}^n) = H^{\ell+2n+1}(\mathbb{R}^n)$. Thus, by Corollary 5.3, the assertion follows from $2n > n$. □

Next, we turn to a variant of the first assertion in Lemma 8.10.

**Lemma 9.4.** *Let $\mu > 0$, $z \in \mathbb{C}$, $\operatorname{Re}(z) > -1$, $R_{1+z}$ given by* (4.6)*, $C$ given by* (6.15)*, and $\mathcal{Q}$ given by* (6.3)*. Then for all $j \in \{1, 2, 3\}$, the integral kernel of*

$$(1-\mu\Delta)^{-1}\operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}\right)(1-\mu\Delta)^{-1}$$

*vanishes on the diagonal, where $\gamma_{1,3}, \gamma_{2,3}, \gamma_{3,3} \in \mathbb{C}^{2\times 2}$ are given as in Remark* 6.1 *(see also Appendix* A*).*

*Proof.* We denote the integral kernel under consideration by $h_j$, $j \in \{1, 2, 3\}$. From Lemma 8.8, one recalls,

$$\begin{aligned}
&(1-\mu\Delta)^{-1}\operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}\right)(1-\mu\Delta)^{-1} \\
&\quad = -(1-\mu\Delta)^{-1}\operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}\Phi\mathcal{Q}R_{1+z}\right)(1-\mu\Delta)^{-1}.
\end{aligned}$$

With $(1-\mu\Delta)^{-1} = (1/\mu)\left((1/\mu)-\Delta\right)^{-1}$ one computes,

$$h_j(x,x) = -\frac{1}{\mu^2}\int_{(\mathbb{R}^3)^3} r_{1/\mu}(x-x_1)\operatorname{tr}_{2d}\Bigg(\gamma_{j,3}\sum_{i_1=1}^{3}\gamma_{i_1,3}(\partial_{i_1}r_{1+z})(x_1-x_2)\Phi(x_2)$$

$$
\begin{aligned}
&\times \sum_{i_2=1}^{3} \gamma_{i_2,3}(\partial_{i_2} r_{1+z})(x_2 - x_3)\Big) r_{1/\mu}(x_3 - x)\, d^n x_1 d^n x_2 d^n x_3 \\
&= \frac{1}{\mu^2} \int_{(\mathbb{R}^3)^3} r_{1/\mu}(x - x_1)\, \mathrm{tr}_{2d} \Big( \gamma_{j,3} \sum_{i_1=1}^{3} \gamma_{i_1,3}(\partial_{i_1} r_{1+z})(x_2 - x_1)\Phi(x_2) \\
&\times \sum_{i_2=1}^{3} \gamma_{i_2,3}(\partial_{i_2} r_{1+z})(x_2 - x_3)\Big) r_{1/\mu}(x_3 - x)\, d^n x_1 d^n x_2 d^n x_3 \\
&= \frac{1}{\mu^2} \int_{(\mathbb{R}^3)^3} r_{1/\mu}(x_1 - x)\, \mathrm{tr}_{2d} \Big( \gamma_{j,3} \sum_{i_1=1}^{3} \gamma_{i_1,3}(\partial_{i_1} r_{1+z})(x_2 - x_1)\Phi(x_2) \\
&\times \sum_{i_2=1}^{3} \gamma_{i_2,3}(\partial_{i_2} r_{1+z})(x_2 - x_3)\Big) r_{1/\mu}(x_3 - x)\, d^n x_1 d^n x_2 d^n x_3, \quad x \in \mathbb{R}^n.
\end{aligned}
$$

The latter expression is symmetric in $x_2$ and $x_3$, by Fubini's theorem. Hence, the assertion follows as in Lemma 8.8 with the help of Corollary A.9. □

Now we are in position to prove the trace theorem for dimension $n = 3$. Of course the principal strategy for the proof is similar to the one for dimensions $n \geqslant 5$ and, thus, need not be repeated.

**Theorem 9.5.** *Let $n = 3$, $L = \mathcal{Q} + \Phi$ given by* (7.1)*, and $\chi_\Lambda$ given by* (7.3)*. Then for all $z \in \mathbb{C}$ with $z \in \varrho(-LL^*) \cap \varrho(-L^*L)$, and $B_L(z)$ given* (7.2)*, $\chi_\Lambda B_L(z)$ is trace class for all $\Lambda > 0$. The limit $f(\cdot) := \lim_{\Lambda\to\infty} \mathrm{tr}(\chi_\Lambda B_L(\cdot))$ exists in the compact open topology and the formula*

$$
\begin{aligned}
f(z) = \frac{i}{16\pi}(1+z)^{-3/2} \lim_{\Lambda\to\infty} &\sum_{j,i_1,i_2=1}^{3} \varepsilon_{j i_1 i_2} \frac{1}{\Lambda} \\
&\times \int_{\Lambda S^2} \mathrm{tr}\left(\Phi(x)\left(\partial_{i_1}\Phi\right)(x)\left(\partial_{i_2}\Phi\right)(x)\right) x_j\, d^{n-1}\sigma(x)
\end{aligned} \tag{9.4}
$$

*holds, where $\varepsilon_{j i_1 i_2}$ denotes the $\varepsilon$-symbol as in Proposition* A.8.

*Proof.* Let $\Lambda > 0$, $\mu > 0$. Denote the integral kernels of $A_L(z)$ and $\sum_j \left[\partial_j, J_L^j(z)\right]$ by $\mathbb{A}_L$ and $\mathbb{J}_L$, respectively, and correspondingly for $A_{L,\mu}(z)$ and $\sum_j \left[\partial_j, J_{L,\mu}^j(z)\right]$, where the respective operators are given by (7.7), (7.6), (9.3), and (9.2). One notes that by Lemma 8.9, $\mathbb{A}_{L,\mu} \to \mathbb{A}_L$ and $\mathbb{J}_{L,\mu} \to \mathbb{J}_L$ pointwise as $\mu \to 0$. One recalls from Proposition 7.4 and Theorem 7.8 together with Proposition 4.3 that (similarly to the case $n = 5$),

$$
2\,\mathrm{tr}(\chi_\Lambda B_L(z)) = \int_{B(0,\Lambda)} \left(\mathbb{A}_L + \mathbb{J}_L\right)(x,x)\, d^n x,
$$

and, as $\mathbb{A}_L$ and the integral kernel of $B_L(z)$ are continuous, so is $\mathbb{J}_L$. Hence, by Lemma 8.9 and using $\mathbb{A}_L(x,x) = 0$ (see Lemma 8.1), one obtains

$$
\begin{aligned}
2\,\mathrm{tr}(\chi_\Lambda B_L(z)) &= \int_{B(0,\Lambda)} \mathbb{A}_L(x,x) + \mathbb{J}_L(x,x)\, d^n x \\
&= \int_{B(0,\Lambda)} \lim_{\mu\to 0} \mathbb{J}_{L,\mu}(x,x)\, d^n x
\end{aligned}
$$

$$= \lim_{\mu\to 0} \int_{B(0,\Lambda)} \mathbb{J}_{L,\mu}(x,x)\, d^n x,$$

where the last equality follows from the fact that the family of integral kernels of

$$\{2B_{L,\mu}(z) - A_{L,\mu}(z)\}_{\mu>0}$$

is locally uniformly bounded: To prove the latter assertion, we note that due to Corollary 5.3, $\{2B_{L,\mu}(z) - A_{L,\mu}(z)\}_{\mu>0}$ defines a uniformly bounded family of continuous linear operators from $H^{\ell}(\mathbb{R}^n)$ (see (5.1)) to $H^{\ell+2n-1}(\mathbb{R}^n)$, $\ell \in \mathbb{R}$. Indeed, this follows from the representation in Lemma 7.7 together with Proposition 5.4 and the fact that for all $s \in \mathbb{R}$, $(1-\mu\Delta)^{-1} \to I$ strongly in $H^s(\mathbb{R}^n)$. Next, we denote

$$\mathbb{K}_{L,\mu} := \left\{x \mapsto g^j_{L,\mu}(z)(x,x)\right\}_{j\in\{1,2,3\}},$$

where $g^j_{L,\mu}(z)$ is the integral kernel of $J^j_{L,\mu}(z)$, $j \in \{1,2,3\}$, and $\mathbb{K}_L$ that for

$$\left\{J^j_L(z) - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})\right\}_{j\in\{1,2,3\}}.$$

Invoking Lemmas 9.4 and 8.9, and hence the fact that $\{x \mapsto \mathbb{K}_{L,\mu}(x)\}_{\mu>0}$ is locally uniformly bounded, one obtains

$$\begin{aligned}
\lim_{\mu\to 0} \int_{B(0,\Lambda)} \mathbb{J}_{L,\mu}(x,x)\, d^n x &= \lim_{\mu\to 0} \int_{\Lambda S^2} \left(\mathbb{K}_{L,\mu}(x), \frac{x}{\Lambda}\right) d^{n-1}\sigma(x) \\
&= \int_{\Lambda S^2} \lim_{\mu\to 0} \left(\mathbb{K}_{L,\mu}(x), \frac{x}{\Lambda}\right) d^{n-1}\sigma(x) \\
&= \int_{\Lambda S^2} \left(\mathbb{K}_{L}(x), \frac{x}{\Lambda}\right) d^{n-1}\sigma(x).
\end{aligned}$$

As in the case $n \geqslant 5$, one computes with the help of Corollary 8.12 that for $z \in \Sigma_{z_0,\vartheta}$, see (8.8), for some fixed $\vartheta \in (0,\pi/2)$ and $z_0 \in \mathbb{R}$ sufficiently large, the limit $\Lambda \to \infty$ actually coincides with $\mathbb{K}_L$ replaced by the vector of integral kernels of $\left\{2\operatorname{tr}_{2d}\left(\gamma_{j,3}\Phi C^2 R^3_{1+z}\right)\right\}_{j\in\{1,2,3\}}$ (employing analogous arguments using Lemmas 8.5, 8.6, and 8.3). Hence one can compute this expression explicitly with the help of Proposition 8.13, ending up with (9.4). $\square$

## 10. The Index Theorem and Some Consequences

Putting the results of the Sections 3 and 7 together, we arrive at the following theorem:

**Theorem 10.1.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $d \in \mathbb{N}$, $\Phi\colon \mathbb{R}^n \to \mathbb{C}^{d\times d}$ admissible (see Definition* 6.11*). Then the operator $L = \mathcal{Q} + \Phi$ given by* (7.1) *is Fredholm and*

$$\begin{aligned} \operatorname{ind}(L) = \left(\frac{i}{8\pi}\right)^{(n-1)/2} &\frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}} \\ &\times \int_{\Lambda S^{n-1}} \operatorname{tr}(\Phi(x)(\partial_{i_1}\Phi)(x)\dots(\partial_{i_{n-1}}\Phi)(x)) x_j \, d^{n-1}\sigma(x), \end{aligned} \tag{10.1}$$

*where $\varepsilon_{ji_1\dots i_{n-1}}$ denotes the $\varepsilon$-symbol as introduced in Proposition* A.8.

*Proof.* Appealing to Theorem 3.4 and Theorem 7.1 (or 9.5), we have $f(0) = \operatorname{ind}(L)$, with $f$ from Theorem 7.1. □

In Corollary 10.11 at the end of this section we will show that actually, $\operatorname{ind}(L) = 0$ for admissible $\Phi$.

Next, we indicate how the index theorem obtained can be generalized to potentials $\Phi$ belonging only to $C_b^2(\mathbb{R}^n; \mathbb{C}^{d\times d})$ satisfying $|\Phi(x)| \geqslant c$ for all $x \in \mathbb{R}^n\backslash B(0,R)$ for some $R > 0$, $c > 0$. More precisely, we will prove the following theorem later in Section 12 in the case where $\Phi$ is $C^\infty$ and in full generality in Section 13:

**Theorem 10.2.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $d \in \mathbb{N}$, $\Phi \in C_b^2(\mathbb{R}^n; \mathbb{C}^{d\times d})$. Assume the following properties*

$$\Phi(x) = \Phi(x)^*, \quad x \in \mathbb{R}^n,$$

*there exists $c > 0$, $R \geqslant 0$ such that $|\Phi(x)| \geqslant c$, $x \in \mathbb{R}^n\backslash B(0,R)$, and that there is $\varepsilon > 1/2$ such that for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| < 3$, there is $\kappa > 0$ such that*

$$\|(\partial^\alpha \Phi)(x)\| \leqslant \begin{cases} \kappa(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa(1+|x|)^{-1-\varepsilon}, & |\alpha| = 2, \end{cases} \quad x \in \mathbb{R}^n.$$

*We recall $\mathcal{Q} = \sum_{j=1}^n \gamma_{j,n}\partial_j$, with $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}$, $j \in \{1,\dots,n\}$, given in* (6.3) *or Theorem* 6.4*. Then the operator $L \coloneqq \mathcal{Q} + \Phi$ considered in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ is a Fredholm operator and*

$$\begin{aligned} \operatorname{ind}(L) = \left(\frac{i}{8\pi}\right)^{(n-1)/2} &\frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}} \\ &\times \int_{\Lambda S^{n-1}} \operatorname{tr}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x)) x_j \, d^{n-1}\sigma(x), \end{aligned} \tag{10.2}$$

*where*

$$U(x) = |\Phi(x)|^{-1}\Phi(x) = \operatorname{sgn}(\Phi(x)), \quad x \in \mathbb{R}.$$

While in this manuscript we focus on the functional analytic proof of Callias' index formula (10.2), we refer to the discussion by Bott and Seeley [14] for its underlying topological setting (homotopy invariants, etc.).

In Theorem 10.2, there are two main difficulties to cope with: on the one hand – in contrast to the situation in Theorem 10.1 – the potential is only assumed to be invertible on the complement of large balls, on the other hand the potential

is only $C^2$. We will address the second case later on, and concern ourselves with the invertibility issue first. However, before providing the proof of Theorem 10.2 in these more general cases, we give a motivating fact underlining the need for Theorem 10.2. In particular, in Theorem 10.6, we show that a particular class of potentials cannot be treated with the help of Theorem 10.1. The main problem preventing the applicability of Theorem 10.1 is the everywhere invertibility assumed in Definition 6.11.

We note that the special case $n = 3$ in connection with Yang–Mills–Higgs fields and monopoles has been discussed in detail in [70, Sect. II.5] and [81, Sect. VIII.4].

Before turning to Theorem 10.6, we shall provide a result that studies the sign of an operator. This study is needed, as the formula in Theorem 10.2 involves $x \mapsto \Phi(x)/|\Phi(x)| = \operatorname{sgn}(\Phi(x))$.

**Theorem 10.3.** *Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H})$, $\operatorname{Re}\big(A^2\big) \geqslant c$ for some $c > 0$. Then the integral (see, e.g.,* [59, Ch. 5, equation (5.3)]*),*

$$\operatorname{sgn}(A) := \frac{2}{\pi} A \int_0^\infty \big(t^2 + A^2\big)^{-1}\, dt$$

*converges in $\mathcal{B}(\mathcal{H})$ and $\operatorname{sgn}(\cdot)$ is analytic on $B\big(A, (\|A\|^2 + c)^{1/2} - \|A\|\big)$. Moreover, if in addition, $A = A^*$ then*

$$\operatorname{sgn}(A) = A|A|^{-1},$$

*and $\operatorname{sgn}(A)$ is unitary.*

*Proof.* From $\operatorname{Re}\big(t^2 + A^2\big) \geqslant t^2 + c$ it follows that $\big\| \big(t^2 + A^2\big)^{-1} \big\| \leqslant (t^2 + c)^{-1}$ and, thus, the integral converges in operator norm. In order to prove analyticity, we first show that given $B \in \mathcal{B}(\mathcal{H})$ with $\operatorname{Re}(B) \geqslant c$ the function $T \mapsto \int_0^\infty \big(t^2 + B + T\big)^{-1}\, dt$ is analytic at 0 with convergence radius at least $c$. Hence, let $B \in \mathcal{B}(\mathcal{H})$ with $\operatorname{Re}(B) \geqslant c$. Then $\big\| \big(t^2 + B\big)^{-1} \big\| \leqslant (t^2 + c)^{-1} \leqslant c^{-1}$ for all $t \in \mathbb{R}$. If $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leqslant \vartheta c$ for some $0 < \vartheta < 1$, then $\Big\| \big(t^2 + B\big)^{-1} T \Big\| \leqslant \vartheta$ for all $t \in \mathbb{R}$ and thus,

$$\begin{aligned}
\int_0^\infty \big(t^2 + (B + T)\big)^{-1}\, dt &= \int_0^\infty \Big(1 + \big(t^2 + B\big)^{-1} T\Big)^{-1} \big(t^2 + B\big)^{-1}\, dt \\
&= \int_0^\infty \sum_{k=0}^\infty \Big(-\big(t^2 + B\big)^{-1} T\Big)^k \big(t^2 + B\big)^{-1}\, dt \\
&= \sum_{k=0}^\infty \int_0^\infty \Big(-\big(t^2 + B\big)^{-1} T\Big)^k \big(t^2 + B\big)^{-1}\, dt.
\end{aligned}$$

One observes that $c_k \colon \mathcal{B}(\mathcal{H})^k \to \mathcal{B}(\mathcal{H})$ given by

$$c_k(T, \dots, T) := \int_0^\infty \Big(-\big(t^2 + B\big)^{-1} T\Big)^k \big(t^2 + B\big)^{-1}\, dt,$$

is a bounded $k$-linear form with bound $c^{-k}\pi/(2\sqrt{c})$. Indeed, for the contractions $T_1, \dots, T_k \in \mathcal{B}(\mathcal{H})$ one estimates

$$\bigg\| \int_0^\infty \prod_{j=1}^k \big[-(t^2 + B)^{-1} T_j\big] (t^2 + B)^{-1}\, dt \bigg\|$$

$$
\begin{aligned}
&\leqslant \int_0^\infty \prod_{j=1}^k \big\| \big[-(t^2+B)^{-1}T_j\big](t^2+B)^{-1}\big\|\,dt \\
&\leqslant \int_0^\infty \big\|(t^2+B)^{-k}\big\|\big\|(t^2+B)^{-1}\big\|\,dt \\
&\leqslant \left(\frac{1}{c}\right)^k \int_0^\infty \frac{1}{t^2+c}\,dt = \left(\frac{1}{c}\right)^k \frac{\pi}{2\sqrt{c}}.
\end{aligned}
$$

In particular, the power series has convergence radius at least $c$. It follows that $T \mapsto \int_0^\infty \big(t^2+T\big)^{-1}\,dt$ is analytic about $A^2$ with convergence radius $c$. If $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| < \big((\|A\|^2+c)^{1/2} - \|A\|\big)$, then

$$
\begin{aligned}
\big\|(A+T)^2 - A^2\big\| &< 2\|A\|\|T\| + \|T\|^2 \\
&\leqslant 2\|A\|\big((\|A\|^2+c)^{1/2} - \|A\|\big) + \big((\|A\|^2+c)^{1/2} - \|A\|\big)^2 \\
&= c.
\end{aligned}
$$

Hence, the map $T \mapsto \int_0^\infty \big(t^2+T^2\big)^{-1}\,dt$ is analytic about $A$ with convergence radius at least $\big((\|A\|^2+c)^{1/2} - \|A\|\big)$.

The equality and unitarity now follow from the functional calculus for self-adjoint opertors and the respective equality for numbers. □

The next fact provides a more detailed account on the behavior of $x \mapsto \operatorname{sgn}(\Phi(x))$ for smooth $\Phi$. We note that the following result has been asserted implicitly in a modified form in [22, last paragraph on p. 226].

**Lemma 10.4.** *Let $n, d \in \mathbb{N}_{\geqslant 1}$, $\Phi \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ pointwise self-adjoint, $c, R \geqslant 0$, $c \neq 0$. Assume that for all $x \in \mathbb{R}^n\backslash B(0,R)$, $|\Phi(x)| \geqslant c$. Let $\tau > 0$. Then there exists $U \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ pointwise self-adjoint, and a function $u \in C_b^\infty(\mathbb{R}^n;\mathbb{R}_{\geqslant 0})$ with $0 \leqslant u \leqslant 1$, $u_{\mathbb{R}^n\backslash B(0,\tau)} = 1$, such that*

$$
U(x) = \operatorname{sgn}(\Phi(x)),\ x \in \mathbb{R}^n\backslash B(0,R) \ \text{ and } \ U(x)^2 = u(x)I_d,\ x \in \mathbb{R}^n.
$$

*Moreover, for all $\beta \in \mathbb{N}_0^n$, $\beta \neq 0$, there exists $\kappa > 0$ such that for all $x \in \mathbb{R}^n\backslash B(0,R)$,*

$$
\|\partial^\beta U(x)\| \leqslant \kappa \sum_{\alpha\in\mathbb{N}_0^n,\gamma\in\mathbb{N},|\alpha|\gamma=|\beta|} \|\partial^\alpha \Phi(x)\|^\gamma.
$$

*Remark* 10.5. $(i)$ We note that the function $U$ constructed in Lemma 10.4 attains values in the set of unitary matrices (on $\mathbb{R}^n\backslash B(0,R)$). Indeed, this follows from Theorem 10.3.

$(ii)$ In the situation of Lemma 10.4, assume, in addition, that $\Phi$ satisfies the following estimates: For some $\varepsilon > 1/2$ and for $\alpha \in \mathbb{N}_0^n$, there is a constant $\kappa_1 > 0$ such that

$$
\|(\partial^\alpha \Phi)(x)\| \leqslant \begin{cases} \kappa_1(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa_1(1+|x|)^{-1-\varepsilon}, & |\alpha| \geqslant 2, \end{cases} \quad x \in \mathbb{R}^n.
$$

Then $U$ constructed in Lemma 10.4 satisfies analogous estimates: For $\alpha \in \mathbb{N}_0^n$ there exists $\kappa_2 > 0$ such that

$$
\|(\partial^\alpha U)(x)\| \leqslant \begin{cases} \kappa_2(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa_2(1+|x|)^{-1-\varepsilon}, & |\alpha| \geqslant 2. \end{cases}
$$

In particular, if $\Phi$ is admissible (see Definition 6.11), then so is $U = \operatorname{sgn}(\Phi)$. ◇

*Proof of Lemma* 10.4. One observes that $x \mapsto \operatorname{sgn}(\Phi(x))$ is $C_b^\infty$ on $\mathbb{R}^n \backslash B(0, R')$ for some $0 < R' < R$, by Theorem 10.3. Moreover, for $j \in \{1, \ldots, n\}$,

$$(\partial_j U)(x) = (\operatorname{sgn}'(\Phi(x)))(\partial_j \Phi)(x).$$

Thus,

$$(\partial_k \partial_j U)(x) = \operatorname{sgn}''(\Phi(x))(\partial_k \Phi)(x)(\partial_j \Phi)(x) + \operatorname{sgn}'(\Phi(x))(\partial_k \partial_j \Phi)(x).$$

Continuing in this manner, we obtain the estimates for the derivatives, once noticing that $x \mapsto \operatorname{sgn}^{(k)}(\Phi(x))$ is bounded for all $k \in \mathbb{N}$. Indeed, by the boundedness of $\Phi$ and since $|\Phi(x)| \geqslant c$ for all $x \in \mathbb{R}^n \backslash B(0, R)$, the set $\{\Phi(x) \,|\, x \in \mathbb{R}^n \backslash B(0, R)\} \subseteq \mathbb{C}^{d \times d}$ is relatively compact and its closure is contained in the domain of analyticity of $\operatorname{sgn}(\cdot)$. Hence, $x \mapsto \operatorname{sgn}^{(k)}(\Phi(x))$ is indeed bounded.

Next, let $\eta \in C_b^\infty(\mathbb{R}^n)$ with

$$\eta(x) \begin{cases} = R', & |x| \leqslant R', \\ \in [R', R], & R' < |x| < R, \\ = |x|, & |x| \geqslant R. \end{cases}$$

Then $\alpha \colon \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n, x \mapsto \eta(|x|)\frac{x}{|x|}$ is $C^\infty$ and $\alpha(x) = x$ for all $|x| \geqslant R$. Let, in addition, $\phi \colon \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be a $C^\infty$-function such that $\phi(x) = 1$ for $x \in \mathbb{R}^n \backslash B(0, \tau)$ and with $0 \leqslant \phi \leqslant 1$ and $\phi(x) = 0$ on $B(0, \tau/2)$. Then a suitable choice for $U$ is

$$x \mapsto \phi(x) \operatorname{sgn}(\Phi(\alpha(x))). \qquad \square$$

One might wonder, whether the function $u$ vanishing at the origin in Lemma 10.4 is really needed. In fact, if it was possible for any arbitrarily differentiable potential $\Phi$ discussed in Theorem 10.2, to choose $u$ in Lemma 10.4 being 1 also at the origin, the only nontrivial assertion of Theorem 10.2 would be the differentiability issue. However, the next example indicates that Theorem 10.2 has a nontrivial application.

**Theorem 10.6.** *Consider the function* $\Phi \colon \mathbb{R}^3 \to \mathbb{C}^{2\times 2}$ *such that*

$$\Phi(x) = \sum_{j=1}^{3} \sigma_j \frac{x_j}{|x|}, \quad |x| \geqslant 1,$$

*as in Example* 4.8. *Then there is no* $U \in C^\infty(\mathbb{R}^3; \mathbb{C}^{2\times 2})$ *with the property that* $U(x) = \Phi(x)$ *for all* $x \in \mathbb{R}^n$, $|x| \geqslant 1$, $U(x) = U(x)^*$ *and for some* $c > 0$, $|U(x)| \geqslant c$, $x \in \mathbb{R}^n$.

*Proof.* We will proceed by contradiction and assume the existence of such a $U$. By Lemma 10.4 (and Remark 10.5 $(i)$), we may assume without loss of generality that $U$ assumes values in the self-adjoint unitary operators in $\mathbb{C}^{2\times 2}$. The latter are of the form

$$W = \begin{pmatrix} a & b + id \\ b - id & c \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}$$

with $W^* W = I_2$. From the latter equation, one reads off

$$\begin{aligned} 1 &= a^2 + b^2 + d^2, \\ 0 &= (a + c)b, \\ 0 &= (a + c)d, \\ 1 &= c^2 + b^2 + d^2. \end{aligned}$$

Hence, either $a \neq -c$, which implies $b = d = 0$ and $a = c = \pm 1$, or $a = -c$ with $a^2 + b^2 + d^2 = 1$. Note that, in the latter case, we have $W = a\sigma_1 + b\sigma_2 + d\sigma_3$ and $\det(W) = -1$. Hence, since $U$ is pointwise invertible everywhere, and $\det(\pm I_2) = 1$, by the intermediate value theorem, one infers that

$$U[\mathbb{R}^3] \subseteq \{a\sigma_1 + b\sigma_2 + c\sigma_3 \,|\, a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1\} =: \mathcal{U}$$

Identifying $\mathcal{U}$ with $S^2$ and using $U|_{S^2} = I_{S^2}$, one observes that $U$ is a retraction of $B(0,1)$ for $S^2$, which is a contradiction. We provide some details for the latter claim. Assume there exists a continuous map $f\colon \overline{B(0,1)} \subset \mathbb{R}^3 \to S^2$ with the property $f(x) = x$ for all $x \in S^2$. Denoting the identity on $\overline{B(0,1)}$ by $I_{\overline{B(0,1)}}$, one considers the homotopy $H$ of $f$ and $I_{\overline{B(0,1)}}$ given by

$$H(\lambda, x) := \lambda f(x) + (1-\lambda) I_{\overline{B(0,1)}}(x), \quad \lambda \in [0,1],\ x \in \overline{B(0,1)}.$$

In the following, we denote by $\deg(g, z_0)$ Brouwer's degree of a function $g\colon \overline{B(0,1)} \to \mathbb{R}^3$ in the point $z_0 \in \mathbb{R}^3 \backslash g[S^2]$. One observes that $0 \in \mathbb{R}^3 \backslash H(\lambda, S^2) = \mathbb{R}^3 \backslash S^2$ for all $\lambda \in [0,1]$, by the hypotheses on $f$. Hence, by homotopy invariance of Brouwer's degree, one gets, using $0 \in I_{\overline{B(0,1)}}[\overline{B(0,1)}]$ and $0 \notin f[\overline{B(0,1)}] = S^2$,

$$1 = \deg(I_{\overline{B(0,1)}}, 0) = \deg(H(0,\cdot), 0) = \deg(H(1,\cdot)) = \deg(f, 0) = 0,$$

a contradiction. $\square$

While we decided to provide an explicit proof of Theorem 10.6, it should be mentioned that is is a special case of "Brouwer's no retraction theorem" (see, e.g., [38, Theorem 3.12]): There is no continuous map $f : \overline{B(0,1)} \to S^{n-1}$ that is the identity on $S^{n-1}$. (Here $\overline{B(0,1)}$ denotes the closed unit ball in $\mathbb{R}^n$, $n \in \mathbb{N}$.)

In the remainder of this section, we study the index formula (10.2) in more detail. More precisely, we will show an invariance principle which will lead to a proof of Corollary 10.11, which shows that for admissible potentials $\Phi$, the index of $\mathcal{Q} + \Phi$ vanishes, reproducing [86, Theorem 5.2] in our context.

Let $n, d \in \mathbb{N}$, $\mathcal{U} \subseteq \mathbb{R}^n$ open, $\Upsilon \in C^1\big(\mathcal{U}; \mathbb{C}^{d\times d}\big)$. For $x \in \mathcal{U}$ we introduce the expression

$$M_\Upsilon(x) := \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\cdots i_n} \operatorname{tr}\big(\partial_{i_1}\Upsilon(x)\cdots\partial_{i_n}\Upsilon(x)\big), \tag{10.3}$$

where $\varepsilon_{i_1\cdots i_n}$ denotes the totally anti-symmetric symbol in $n$ coordinates.

*Remark* 10.7. The relationship of the index formula for potentials $\Phi$ as in Theorem 10.2 and the function defined in (10.3) is as follows: Let $U$ be $C^2$-smooth with $U = \operatorname{sgn}(\Phi)$ on the complement of a sufficiently large ball. For $\Lambda > 0$, one computes with the help of Gauss' divergence theorem

$$\begin{aligned}
&\frac{1}{\Lambda}\sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\dots i_n} \int_{\Lambda S^{n-1}} \operatorname{tr}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x))x_{i_n}\, d^{n-1}\sigma(x)\\
&\quad= \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\dots i_n} \int_{B(0,\Lambda)} \operatorname{tr}((\partial_{i_1}U)(x)\dots(\partial_{i_n}U)(x))\, d^n x\\
&\quad= \int_{B(0,\Lambda)} M_U(x) d^n x.
\end{aligned}$$

Hence, the index formula for the operator $L = \mathcal{Q} + \Phi$ discussed in Theorem 10.2 may be rewritten as follows

$$\operatorname{ind}(L) = \left(\frac{i}{8\pi}\right)^{(n-1)/2} \frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2} \int_{B(0,\Lambda)} M_U(x)\, d^n x. \tag{10.4}$$

$\diamond$

**Definition 10.8** (Transformations of constant orientation)**.** *Let $n \in \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^n$ open, dense. We say that $T\colon \mathcal{U} \to \mathbb{R}^n$ is a* transformation of constant orientation*, if the following properties (i)–(iii) are satisfied:*

*(i) $T$ is continuously differentiable and injective.*

*(ii) $T[\mathcal{U}]$ is dense in $\mathbb{R}^n$.*

*(iii) The function $\mathcal{U} \ni x \mapsto \operatorname{sgn}(\det(T'(x)))$ is either identically $1$ or $-1$. We define $\operatorname{sgn}(T) \coloneqq \operatorname{sgn}(\det(T'(x)))$ for some (and hence for all) $x \in \mathcal{U}$.*

The sought after invariance principle then reads as follows:

**Theorem 10.9.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $d \in \mathbb{N}$, $\Phi \in C_b^2\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$. Assume the following properties:*

$$\Phi(x) = \Phi(x)^*, \quad x \in \mathbb{R}^n,$$

*there exists $c > 0$, $R \geqslant 0$ such that $|\Phi(x)| \geqslant c$ for all $x \in \mathbb{R}^n \backslash B(0,R)$, and that there is $\varepsilon > 1/2$ such that for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| < 3$, there is $\kappa > 0$ such that*

$$\|(\partial^\alpha \Phi)(x)\| \leqslant \begin{cases} \kappa(1+|x|)^{-1}, & |\alpha| = 1, \\ \kappa(1+|x|)^{-1-\varepsilon}, & |\alpha| = 2, \end{cases} \quad x \in \mathbb{R}^n.$$

*We recall $\mathcal{Q} = \sum_{j=1}^n \gamma_{j,n} \partial_j$, with $\gamma_{j,n} \in \mathbb{C}^{2^{\widehat{n}} \times 2^{\widehat{n}}}$, $j \in \{1, \dots, n\}$, given in (6.3) or in Theorem 6.4. In addition, let $T\colon \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ (with $\mathcal{U}$ as in Definition 10.8) be a transformation of constant orientation. Assume that $\Phi_T \coloneqq \overline{\Phi \circ T}$ (the closure of the mapping $\Phi \circ T$) satisfies the assumptions imposed on $\Phi$. Then $L_1 = \mathcal{Q} + \Phi$ and $L_2 = \mathcal{Q} + \Phi_T$ are Fredholm and*

$$\operatorname{ind}(L_1) = \operatorname{sgn}(T) \operatorname{ind}(L_2).$$

Before proving Theorem 10.9 we need a chain rule for the function defined in (10.3).

**Lemma 10.10.** *Let $n, d \in \mathbb{N}$, $\mathcal{U} \subseteq \mathbb{R}^n$ open, $\Phi \in C^1\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$, $T \in C^1(\mathcal{U}; \mathbb{R}^n)$. Then,*

$$M_{\Phi\circ T}(x) = M_\Phi(T(x)) \det(T'(x)), \quad x \in \mathcal{U}.$$

*Proof.* One recalls that for an $n \times n$-matrix $A = (a_{ij})_{i,j\in\{1,\dots,n\}} \in \mathbb{C}^{n\times n}$, its determinant may be computed as follows

$$\det(A) = \sum_{i_1,\dots,i_n=1}^n \varepsilon_{i_1\cdots i_n} a_{i_1 1} \cdots a_{i_n n}.$$

Consequently, for $k_1, \dots, k_n \in \{1, \dots, n\}$, one gets

$$\varepsilon_{k_1\cdots k_n} \det(A) = \sum_{i_1,\dots,i_n=1}^n \varepsilon_{i_1\cdots i_n} a_{i_1 k_1} \cdots a_{i_n k_n}.$$

Using the chain rule of differentiation, one obtains for $x \in \mathcal{U}$,

$$\begin{aligned}
M_{\Phi\circ T}(x) &= \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\cdots i_n} \operatorname{tr}_d \big(\partial_{i_1}(\Phi\circ T)(x)\cdots\partial_{i_n}(\Phi\circ T)(x)\big)\\
&= \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\cdots i_n} \operatorname{tr}_d \Big( \sum_{k_1=1}^{n} \partial_{k_1}\Phi(T(x))\partial_{i_1}T_{k_1}(x)\cdots \sum_{k_n=1}^{n}\partial_{k_n}\Phi(T(x))\partial_{i_n}T_{k_n}(x)\Big)\\
&= \sum_{k_1,\dots,k_n=1}^{n}\sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\cdots i_n}\partial_{i_1}T_{k_1}(x)\cdots\partial_{i_n}T_{k_n}(x)\\
&\qquad\qquad \times \operatorname{tr}_d\big(\partial_{k_1}\Phi(T(x))\cdots\partial_{k_n}\Phi(T(x))\big)\\
&= \sum_{k_1,\dots,k_n=1}^{n} \varepsilon_{k_1\cdots k_n}\det(T'(x))\operatorname{tr}_d\big(\partial_{k_1}\Phi(T(x))\cdots\partial_{k_n}\Phi(T(x))\big)\\
&= M_\Phi(T(x))\det(T'(x)). \qquad \square
\end{aligned}$$

*Proof of Theorem* 10.9. Let $U$ be $C^2$-smooth and such that $\operatorname{sgn}(\Phi) = U$ on complements of sufficiently large balls. One observes that $U_T := \overline{U\circ T} = \operatorname{sgn}(\Phi_T)$. In particular, $\operatorname{ind}(\mathcal{Q}+\Phi_T) = \operatorname{ind}(\mathcal{Q}+U_T)$. Next, we set

$$c_n := \frac{1}{2}\left(\frac{i}{8\pi}\right)^{(n-1)/2}\frac{1}{[(n-1)/2]!}.$$

By Theorem 10.2 together with Remark 10.7, and taking into account the chain rule, Lemma 10.10, one computes,

$$\begin{aligned}
\operatorname{ind}(\mathcal{Q}+U_T) &= c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)} M_{U_T}(x)\,d^nx\\
&= c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)} M_{U\circ T}(x)\,d^nx\\
&= c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)} M_U(T(x))\det(T'(x))\,d^nx\\
&= \operatorname{sgn}(T)c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)} M_U(T(x))|\det(T'(x))|\,d^nx\\
&= \operatorname{sgn}(T)c_n \lim_{\Lambda\to\infty}\int_{T[B(0,\Lambda)]} M_U(x)\,d^nx\\
&= \operatorname{sgn}(T)c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)\cap T[B(0,\Lambda)]} M_U(x)\,d^nx,
\end{aligned}$$

using the transformation rule for integrals.

To conclude the proof, we are left with showing

$$c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)\cap T[B(0,\Lambda)]} M_U(x)\,d^nx = \operatorname{ind}(\mathcal{Q}+U).$$

For this purpose one notes that $T$ is continuously invertible, by hypothesis. Hence, the range of $T$ is open. Since the range of $T$ is also dense, $\{\chi_{T[B(0,\Lambda)]}\}_{\Lambda\in\mathbb{N}}$ converges in the strong operator topology of $\mathcal{B}\big(L^2(\mathbb{R}^n)\big)$ to $I_{L^2(\mathbb{R}^n)}$, where $\chi_{T[B(0,\Lambda)]}$ denotes the characteristic function of the set $T[B(0,\Lambda)]$, $\Lambda > 0$. Thus, for $L = \mathcal{Q}+U$, one

computes

$$\begin{aligned}\operatorname{ind}(L) &= \lim_{\Lambda\to\infty}\lim_{z\to 0_+} z\,\operatorname{tr}_{L^2(\mathbb{R}^n)}\big(\chi_{T[B(0,\Lambda)]}\chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}d}\big((L^*L+z)^{-1}-(LL^*+z)^{-1}\big)\big)\\ &= c_n \lim_{\Lambda\to\infty}\int_{B(0,\Lambda)\cap T[B(0,\Lambda)]} M_U(x)\,d^n x,\end{aligned}$$

proving the assertion. □

Finally, we apply Theorem 10.9 and prove that for admissible potentials $\Phi$, $\operatorname{ind}(\mathcal{Q}+\Phi)=0$:

**Corollary 10.11.** *Let $n\in\mathbb{N}_{\geqslant 3}$ odd, $d\in\mathbb{N}$. Let $\Phi$ be admissible, see Definition* 6.11*. Let $\mathcal{Q}$ be as in* (6.3) *and $L=\mathcal{Q}+\Phi$ as in* (7.1)*. Then $L$ is Fredholm and* $\operatorname{ind}(L)=0$.

*Proof.* By invariance of the Fredholm index under relatively compact perturbations (cf. Theorem 3.6 (*iii*)), we can assume without loss of generality, that $\Phi$ is constant in a neighborhood of 0. We consider $T\colon \mathbb{R}^n\backslash\{0\}\to\mathbb{R}^n$ given by

$$T(x) := \frac{x}{|x|^2}, \quad x\in\mathbb{R}^n\backslash\{0\}.$$

One observes that $T$ is a transformation of constant orientation. Moreover, as $\Phi$ is admissible, so is $\Phi_T := \overline{\Phi\circ T}$. In particular, since $\Phi$ is constant in a neighborhood of 0, we find $\Lambda>0$ such that for all $x\in\mathbb{R}^n$ with $|x|\geqslant\Lambda$, $(\partial_i\Phi_T)(x)=0$. Hence, $\operatorname{ind}(\mathcal{Q}+\Phi_T)=0$, by Theorem 10.1 and, thus $\operatorname{ind}(\mathcal{Q}+\Phi)=0$, by Theorem 10.9. □

For an entirely diffferent approach to Corollary 10.11 we refer again to [86, Theorem 5.2].

*Remark* 10.12. As kindly pointed out to us by one of the referees, Corollary 10.11 permits a more elementary proof as follows. If $\Phi$ is admissible, then $x\mapsto\Phi_t(x):=\Phi(tx)$, $t>0$, is also admissible and the associated operators $L_t\colon H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}\to L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$, $t>0$, are all Fredholm. In addition, the map, $(0,\infty)\ni t\mapsto L_t\in\mathcal{B}\big(H^1(\mathbb{R}^n)^{2^{\widehat{n}}d}, L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}\big)$ is continuous. Thus (cf. Corollary 3.7),

$$\operatorname{ind}(L_1)=\operatorname{ind}(L_t), \quad t>0. \tag{10.5}$$

However, (6.14) leads to

$$L_t^*L_t = -\Delta I_{2^{\widehat{n}}d} - C_t + \Phi_t^2, \quad L_tL_t^* = -\Delta I_{2^{\widehat{n}}d} + C_t + \Phi_t^2, \tag{10.6}$$

where

$$C_t = \sum_{j=1}^n \gamma_{j,n}(\partial_j\Phi_t) = (\mathcal{Q}\Phi_t), \quad t>0. \tag{10.7}$$

Hence, for some constant $c>0$, $\|C_t\|_{\mathcal{B}(L^2(\mathbb{R}^n)^{2^{\widehat{n}}d})}\leqslant c\,t$ for $0<t$ sufficiently small. In particular, for $0<t$ sufficiently small, the operators $L_t^*L_t$ and $L_tL_t^*$ are boundedly invertible and hence $\operatorname{ind}(L_t)=0$, implying $\operatorname{ind}(L_1)=\operatorname{ind}(L)=0$. ⋄

## 11. Perturbation Theory for the Helmholtz Equation

Before we are in a position to provide a proof of Theorem 10.2, we need some results concerning the perturbation theory of Helmholtz operators. More precisely, we study operators (and their fundamental solutions) of the form

$$(-\Delta + \mu + \eta)$$

in odd space dimensions $n \geqslant 3$ and $\eta \in L^\infty(\mathbb{R}^n)$ with small support around the origin and $\mu \in \mathbb{C}_{\mathrm{Re}>0}$. For $\mu \in \mathbb{C}_{\mathrm{Re}>0}$, $\eta \in L^\infty(\mathbb{R}^n)$, recalling $R_\mu = (-\Delta + \mu)^{-1}$, one formally computes

$$\begin{aligned} R_{\eta+\mu} &:= \big( -\Delta + \eta + \mu)\big)^{-1} = \big((-\Delta + \mu)(1 + R_\mu \eta\big)^{-1} \\ &= \sum_{k=0}^{\infty} (R_\mu(-\eta))^k R_\mu. \end{aligned} \tag{11.1}$$

This computation can be made rigorous, if $\|R_\mu(\eta)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} < 1$. The first aim of this section is to provide a proof of the fact that if $\|\eta\|_{L^\infty} \leqslant 1$, then indeed $\|R_\mu(\eta)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} < 1$ for "sufficiently" many $\mu$, that is, for $\mu$ belonging to the closed sector

$$\overline{\Sigma_{\mu_0,\vartheta}} = \{z \in \mathbb{C} \,|\, \mathrm{Re}(\mu) \geqslant \mu_0, |\arg(\mu)| \leqslant \vartheta\} \tag{11.2}$$

for some $\mu_0 \in \mathbb{R}$, $\vartheta \in [0, \frac{\pi}{2}]$, provided the support of $\eta$ is sufficiently small.

For $\mu > 0$, $x, y \in \mathbb{R}^n$, $x \neq y$, we introduce

$$s_\mu(x-y) := \frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|^{n-2}}. \tag{11.3}$$

The next lemma shows that the Helmholtz Green's function basically behaves like $s_\mu$ in (11.3). We note that a similar estimate was used in [22, p. 224, formula (c)]. However, we further remark that the factor $\lambda$ introduced in the following result does not occur in [22, p. 224, formula (c)], yielding a hidden $z$-dependence of the constant $K$ occuring there.

**Lemma 11.1.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $\lambda \in (0,1)$. For $\mu > 0$ denote the integral kernel of $R_\mu = (-\Delta + \mu)^{-1}$ in $L^2(\mathbb{R}^n)$ by $r_\mu$, see Lemma* 5.11 *or* (5.11)*, and let $s_\mu$ be as in* (11.3)*. Then there exist $c_1, c_2 > 0$ such that for all $\mu > 0$,*

$$r_\mu(x-y) \leqslant c_1 s_{\lambda\mu}(x-y), \text{ and } s_\mu(x-y) \leqslant c_2 r_\mu(x-y), \quad x, y \in \mathbb{R}^n,\ x \neq y.$$

*Proof.* For the first inequality, one observes that for $k \in \{0, \dots, \widehat{n}-1\}$, with $n = 2\widehat{n}+1$, the function

$$\mathbb{R}_{\geqslant 0} \ni \beta \mapsto \beta^k e^{-(1-\sqrt{\lambda})\beta}$$

is bounded by some $d_k > 0$. Next, let $x, y \in \mathbb{R}^n$, $x \neq y$ and $r := |x-y|$, $\mu > 0$. Then, for $k \in \{0, \dots, \widehat{n}-1\}$, one estimates

$$\frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|^{n-2-k}} (\sqrt{\mu})^k = \frac{e^{-\sqrt{\mu} r}}{r^{n-2}} (r\sqrt{\mu})^k \leqslant d_k \frac{e^{-\sqrt{\lambda\mu} r}}{r^{n-2}}.$$

Hence, the first inequality asserted follows from Lemma 5.11. Employing again Lemma 5.11, the second inequality can be derived easily. □

We can now come to the announced result of bounding the operator norm of $R_\mu \eta$ given $\eta$ is supported on a small set. We note that smallness of the support is independent of $\mu$, if one assumes $\mu$ to lie in a sector.

**Lemma 11.2.** *Let $\mu_0 > 0$, $\vartheta \in (0, \pi/2)$, $\beta > 0$, $n \in \mathbb{N}_{\geqslant 3}$ odd. Then there exists $\tau > 0$ such that for all $\mu \in \Sigma_{\mu_0,\vartheta}$, see* (11.2),

$$\|R_\mu \eta\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leqslant \beta$$

*for all $\eta \in L^\infty(\mathbb{R}^n)$, $\|\eta\|_{L^\infty} \leqslant 1$ and* $\operatorname{supp}(\eta) \subset B(0,\tau)$.

*Proof.* Let $\tau > 0$ and $\eta \in L^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\eta) \subset B(0,\tau)$ and $\|\eta\|_{L^\infty} \leqslant 1$. Let $\mu \in \Sigma_{\mu_0,\vartheta}$ and denote the fundamental solution of $(-\Delta + \mu)$ by $r_\mu$, see also Lemma 5.12. By estimate (5.15) in Lemma 5.12, there exists $c_1 \geqslant 1$ such that

$$|r_\mu(x-y)| \leqslant c_1 r_{\mathrm{Re}\mu}(x-y), \quad x, y \in \mathbb{R}^n,\ x \neq y,\ \mu \in \Sigma_{\mu_0,\vartheta}.$$

Next, by Lemma 11.1, there exists $c_2 > 0$ such that for all $\mu \geqslant \mu_0$,

$$r_\mu(x-y) \leqslant c_2 s_{\frac{1}{2}\mu}(x-y), \quad x, y \in \mathbb{R}^n,\ x \neq y.$$

For $\mu \geqslant \mu_0$, one notes that $\|s_{\mu/2}\|_{L^1(\mathbb{R}^n)} \leqslant \|s_{\mu_0/2}\|_{L^1(\mathbb{R}^n)} < \infty$. Hence, for $\mu \in \Sigma_{\mu_0,\vartheta}$ and $u \in C_0^\infty(\mathbb{R}^n)$, one gets

$$\begin{aligned}
\|R_\mu(\eta)u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} r_\mu(x-y)\eta(y)u(y) d^n y \right|^2 d^n x \\
&= \int_{\mathbb{R}^n} \left| \int_{B(0,\tau)} r_\mu(x-y)\eta(y)u(y) d^n y \right|^2 d^n x \\
&\leqslant c_1^2 \int_{\mathbb{R}^n} \left( \int_{B(0,\tau)} r_{\mathrm{Re}(\mu)}(x-y)|\eta(y)||u(y)| d^n y \right)^2 d^n x \\
&\leqslant c_1^2 c_2^2 \int_{\mathbb{R}^n} \left( \int_{B(0,\tau)} s_{\frac{1}{2}\mathrm{Re}(\mu)}(x-y) d^n y \right) \int_{B(0,\tau)} s_{\frac{1}{2}\mathrm{Re}(\mu)}(x-y)|u(y)|^2 \, d^n y d^n x \\
&\leqslant c_1^2 c_2^2 \int_{\mathbb{R}^n} \left( \int_{B(0,\tau)} s_{\frac{1}{2}\mathrm{Re}(\mu)}(y) d^n y \right) \int_{\mathbb{R}^n} s_{\frac{1}{2}\mathrm{Re}(\mu)}(x-y)|u(y)|^2 \, d^n y d^n x \\
&= c_1^2 c_2^2 \int_{B(0,\tau)} s_{\frac{1}{2}\mathrm{Re}(\mu)}(y) \, d^n y \, \|s_{\frac{1}{2}\mu_0}\|_{L^1(\mathbb{R}^n)} \|u\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

One observes that

$$\int_{B(0,\tau)} s_{\frac{1}{2}\mathrm{Re}(\mu)}(y) \, d^n y \leqslant \omega_{n-1} \int_0^\tau \frac{1}{r^{n-2}} r^{n-1} \, dr = \frac{\tau^2}{2\omega_{n-1}},$$

and hence,

$$\|R_\mu \eta\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leqslant c_1 c_2 \sqrt{\frac{\|s_{\mu_0/2}\|_{L^1(\mathbb{R}^n)}}{2\omega_{n-1}}} \tau.$$ ☐

*Remark* 11.3. (*i*) Let $\mu_0 > 0$, and $\vartheta \in (0, \frac{\pi}{2})$, $\kappa > 0$. Then for all $\mu \in \Sigma_{\mu_0,\vartheta}$ (see (11.2)), there exists $\tau > 0$ such that for $\eta \in L^\infty(\mathbb{R}^n)$, with $\|\eta\|_{L^\infty} \leqslant \kappa$ and $\eta = 0$ on $\mathbb{R}^n \backslash B(0,\tau)$, the operator $R_{\eta+\mu} = (-\Delta + \eta + \mu)^{-1}$ exists as a bounded linear operator in $L^2(\mathbb{R}^n)$ and its norm is arbitarily close to $\|R_\mu\|$. Indeed, for $\beta < 1$ with $\|R_\mu \eta\| \leqslant \beta$ one computes

$$\|R_{\eta+\mu}\| \leqslant \sum_{k=0}^\infty \beta^k \|R_\mu\| = \frac{1}{1-\beta}\|R_\mu\|.$$

(*ii*) In the situation of part (*i*), we shall now elaborate some more on the properties of $R_{\eta+\mu}$ with $\|R_\mu \eta\| \leqslant \beta < 1$ for all $\mu \in \Sigma_{\mu_0,\vartheta}$. Assuming, in addition, $\eta \in C^\infty$,

then $R_{\eta+\mu}$ extends by interpolation to a bounded linear operator to the full Sobolev scale $H^s(\mathbb{R}^n)$ (see (5.1) for a definition), $s \in \mathbb{R}$. Moreover, from

$$(-\Delta+\mu)R_{\eta+\mu} = (-\Delta+\mu)\sum_{k=0}^{\infty} R_\mu(-\eta R_\mu)^k = \sum_{k=0}^{\infty}((-\eta)R_\mu)^k,$$

one gets $\|(-\Delta+\mu)R_{\eta+\mu}\| \leqslant (1-\beta)^{-1}$, yielding $R_{\eta+\mu} \in \mathcal{B}(H^s(\mathbb{R}^n); H^{s+2}(\mathbb{R}^n))$ for all $s \in \mathbb{R}$. $\diamond$

With Lemma 11.2 we have an a priori condition on the support of $\eta$ to make the operator $R_{\eta+\mu}$ well-defined. The forthcoming results, the very reason of this entire section, provide estimates for the integral kernels of the perturbed operator in terms of the unperturbed one. Of course, these estimates also rely on a Neumann series type argument. The main step is the following lemma.

**Lemma 11.4.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $\mu_0 > 0$, $\kappa > 0$. For any $\lambda \in (0,1)$, there exists $\tau > 0$ such that for all $\eta \in L^\infty(\mathbb{R}^n)$, with $\|\eta\|_{L^\infty} \leqslant \kappa$ and $\operatorname{supp}(\eta) \subset B(0,\tau)$, such that for all $k \in \mathbb{N}_{\geqslant 3}$, $\mu \geqslant \mu_0$, the integral kernel $\widetilde{r}_k$ of $(R_\mu \eta)^k R_\mu$ satisfies*

$$|\widetilde{r}_k(x,y)| \leqslant \lambda^k r_{\mu/4}(x-y), \quad x,y \in \mathbb{R}^n,\ x \neq y,$$

*where $r_\mu$ is the integral kernel of $R_\mu = (-\Delta+\mu)^{-1}$ given by* (5.11).

We postpone the proof of Lemma 11.4 and show three preparatory results first.

**Lemma 11.5.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $\mu > 0$, $\tau > 0$, $s_\mu$ as in* (11.3). *Then for all $x,z \in \mathbb{R}^n$, $x \neq z$, the inequality,*

$$\int_{B(0,\tau)} s_\mu(x-y)s_\mu(y-z)\,d^n y \leqslant 2^{n-3}\omega_{n-1}\tau^2 s_\mu(x-z),$$

*holds, with $\omega_{n-1}$ the $(n-1)$-dimensional volume of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ (see also* (5.6)*).*

*Proof.* One notes that, by the triangle inequality,

$$e^{-\sqrt{\mu}|x-y|}e^{-\sqrt{\mu}|y-z|} \leqslant e^{-\sqrt{\mu}|x-z|}, \quad x,y,z \in \mathbb{R}^n.$$

Hence, one is left with showing

$$\int_{B(0,\tau)} \frac{1}{|x-y|^{n-2}}\frac{1}{|y-z|^{n-2}}\,d^n y \leqslant 2^{n-3}\omega_{n-1}\tau^2\frac{1}{|x-z|^{n-2}}, \quad x,y,z \in \mathbb{R}^n,\ x \neq z.$$

Let $x,y,z \in \mathbb{R}^n$. Then

$$|x-z|^{n-2} \leqslant (|x-y|+|y-z|)^{n-2} \leqslant 2^{n-3}(|x-y|^{n-2}+|y-z|^{n-2}).$$

Hence,

$$\begin{aligned}
&\int_{B(0,\tau)} \frac{|x-z|^{n-2}}{|x-y|^{n-2}|y-z|^{n-2}}\,d^n y \\
&\quad \leqslant \int_{B(0,\tau)} \frac{2^{n-3}(|x-y|^{n-2}+|y-z|^{n-2})}{|x-y|^{n-2}|y-z|^{n-2}}\,d^n y \\
&\quad = 2^{n-3}\int_{B(0,\tau)} \left(\frac{1}{|x-y|^{n-2}}+\frac{1}{|y-z|^{n-2}}\right)d^n y \\
&\quad \leqslant 2^{n-2}\int_{B(0,\tau)} \frac{1}{|y|^{n-2}}\,d^n y
\end{aligned}$$

$$= 2^{n-2}\omega_{n-1}\int_0^\tau r\,dr = 2^{n-3}\omega_{n-1}\tau^2. \qquad \square$$

**Proposition 11.6.** *Let $n \in \mathbb{N}$, $\mu_0 > 0$, and $q \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n \backslash \{0\})$. Assume that $V_q$, the operator defined by convolution with $q$, defines a self-adjoint, nonnegative operator in $L^2(\mathbb{R}^n)$. Then for all $\mu \geqslant \mu_0$ and $x, y \in \mathbb{R}^n$, $x \neq y$,*

$$\mu_0 \int_{\mathbb{R}^n} r_\mu(x - x_1) q(x_1 - y)\, d^n x_1 \leqslant q(x - y).$$

*Proof.* Let $\mu \geqslant \mu_0$. As the convolution with $q$ commutes with differentiation, it also commutes with $(-\Delta + \mu)$, $R_\mu$ or powers thereof. Since $V_q \geqslant 0$, there exists a unique nonnegative square root $V_q^{1/2}$, which also commutes with $R_\mu$, $R_\mu^{-1}$ and powers thereof. For $\phi \in H^2(\mathbb{R}^n)$ and $\mu \geqslant \mu_0$,

$$\begin{aligned}\big((-\Delta + \mu)\phi, V_q\phi\big)_{L^2} &= \big((-\Delta + \mu)V_q^{1/2}\phi, V_q^{1/2}\phi\big)_{L^2} \\ &\geqslant \mu_0\big(V_q^{1/2}\phi, V_q^{1/2}\phi\big)_{L^2} \\ &= \mu_0\big(\phi, V_q\phi\big)_{L^2}.\end{aligned}$$

Putting $\phi \coloneqq (-\Delta + \mu)^{-1/2}\psi = R_\mu^{1/2}\psi$ for some $\psi \in H^2(\mathbb{R}^n)$, one infers

$$\begin{aligned}\big(\psi, V_q\psi\big)_{L^2} &\geqslant \mu_0\big(R_\mu^{1/2}\psi, V_q R_\mu^{1/2}\psi\big)_{L^2} \\ &\geqslant \mu_0\big(\psi, R_\mu V_q\psi\big)_{L^2}.\end{aligned}$$

As $(-\Delta + \mu)^{-1/2}[H^2(\mathbb{R}^n)]$ is dense in $L^2(\mathbb{R}^n)$, it follows that $V_q - \mu_0 R_\mu V_q$ is a nonnegative integral operator, which implies the asserted inequality. $\square$

Applying Proposition 11.6 with $q = r_\mu$ twice, one gets the proof of the following result.

**Lemma 11.7.** *Let $n \in \mathbb{N}$, $\mu_0 > 0$. Then for all $\mu \geqslant \mu_0$,*

$$\mu_0^2 \int_{(\mathbb{R}^n)^2} r_\mu(x - x_1) r_\mu(x_1 - x_2) r_\mu(x_2 - y)\, d^n x_1 d^n x_2 \leqslant r_\mu(x - y), \quad x, y \in \mathbb{R}^n,\ x \neq y. \tag{11.4}$$

*Proof of Lemma* 11.4. Recalling (11.3),

$$s_\mu(x - y) = \frac{e^{-\sqrt{\mu}|x-y|}}{|x - y|^{n-2}}, \quad x, y \in \mathbb{R}^n,\ x \neq y,\ \mu > 0.$$

By Lemma 11.1, there exist $c_1, c_2 > 0$ such that for all $\mu > 0$ and $x, y \in \mathbb{R}^n$, $x \neq y$,

$$r_\mu(x - y) \leqslant c_1 s_{\mu/4}(x - y), \quad s_{\mu/4}(x - y) \leqslant c_2 r_{\mu/4}(x - y). \tag{11.5}$$

Next, one recalls, with $n = 2\widehat{n} + 1$ for some $\widehat{n} \in \mathbb{N}$, from Lemma 5.12, equation (5.16), that

$$r_\mu(x - y) \leqslant 2^{\widehat{n}-1} r_{\mu/4}(x - y), \quad x, y \in \mathbb{R}^n,\ x \neq y.$$

Let $\tau > 0$. We estimate for $x, y \in \mathbb{R}^n$, $x \neq y$, $\eta \in L^\infty(\mathbb{R}^n)$, with $\operatorname{supp}(\eta) \subset B(0, \tau)$, $\mu \geqslant \mu_0$, using Lemma 11.5 and inequality (11.5), with $\kappa_\tau \coloneqq 2^{n-3}\omega_{n-1}\tau^2$,

$$\begin{aligned}|\widetilde{r}_k(x, y)| &= \bigg| \int_{(\mathbb{R}^n)^k} r_\mu(x - x_1)\eta(x_1) r_\mu(x_1 - x_2) \cdots \eta(x_k) r_\mu(x_k - y)\, d^n x_1 \cdots d^n x_k \bigg| \\ &\leqslant \|\eta\|_{L^\infty}^k \int_{(B(0,\tau))^k} r_\mu(x - x_1) r_\mu(x_1 - x_2) \cdots r_\mu(x_k - y)\, d^n x_1 \cdots d^n x_k\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|\eta\|_{L^\infty}^k c_1^{k-1} \int_{(B(0,\tau))^k} r_\mu(x-x_1)s_{\mu/4}(x_1-x_2) \\
&\qquad\qquad \cdots \times s_{\mu/4}(x_{k-1}-x_k)r_\mu(x_k-y)\,d^nx_1\cdots d^nx_k \\
&\leqslant \|\eta\|_{L^\infty}^k (c_1\kappa_\tau)^{k-1} \int_{(B(0,\tau))^2} r_\mu(x-x_1)s_{\mu/4}(x_1-x_k)r_\mu(x_k-y)\,d^nx_1d^nx_k \\
&\leqslant \|\eta\|_{L^\infty}^k (c_1\kappa_\tau)^{k-1} c_2 \int_{(B(0,\tau))^2} r_\mu(x-x_1)r_{\mu/4}(x_1-x_k)r_\mu(x_k-y)\,d^nx_1d^nx_k \\
&\leqslant \|\eta\|_{L^\infty}^k (c_1\kappa_\tau)^{k-1} 2^{n-3} c_2 \int_{(\mathbb{R}^n)^2} r_{\mu/4}(x-x_1)r_{\mu/4}(x_1-x_k) \\
&\qquad\qquad \times r_{\mu/4}(x_k-y)\,d^nx_1d^nx_k \\
&\leqslant \|\eta\|_{L^\infty}^k \frac{16}{\mu_0^2}(c_1\kappa_\tau)^{k-1}2^{n-3}c_2 r_{\mu/4}(x-y),
\end{aligned}
$$

where, in the last estimate, we used Lemma 11.7. □

Having proved Lemma 11.4, we can now formulate and prove the result for the estimate of the perturbed and the unperturbed integral kernels (Green's functions).

**Theorem 11.8.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $\mu_0 > 0$, $\vartheta \in (0,\pi/2)$, $\kappa > 0$. Then there exists $c, \tau > 0$ such that for all $\mu \in \Sigma_{\mu_0,\vartheta}$ and $\eta \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta) \subset B(0,\tau)$, $\|\eta\|_{L^\infty} \leqslant \kappa$, the estimate*

$$
|r_{\eta+\mu}(x,y)| \leqslant c r_{\mathrm{Re}(\mu)}(x-y), \quad x,y \in \mathbb{R}^n,\ x \neq y,
$$

*holds, where $r_{\eta+\mu}$ and $r_{\mathrm{Re}(\mu)}$ are the integral kernels for the operators $R_{\eta+\mu} = (-\Delta+\eta+\mu)^{-1}$ and $R_{\mathrm{Re}(\mu)}$, respectively.*

*Proof.* One recalls that $r_\mu$ denotes the integral kernel of $R_\mu$. According to Lemma 5.12, (5.15), there exists $c_1 \geqslant 1$ such that for all $\mu \in \Sigma_{\mu_0,\vartheta}$ one has

$$
|r_\mu(x-y)| \leqslant c_1 r_{\mathrm{Re}(\mu)}(x-y), \quad x,y \in \mathbb{R}^n,\ x \neq y.
$$

Next, by Lemma 11.2, one chooses $\tau_1 > 0$ such that $\|R_\mu\eta\| \leqslant 1/2$ for all $\mu \in \Sigma_{\mu_0,\vartheta}$ and $\eta \in L^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta) \subset B(0,\tau_1)$ and $\|\eta\|_{L^\infty} \leqslant \kappa$, implying that $R_{\eta+\mu}$ is a well-defined bounded linear operator in $L^2(\mathbb{R}^n)$ (see, e.g., Remark 11.3).

Let $\tau_2 > 0$ be such that for all $k \in \mathbb{N}_{\geqslant 1}$, the integral kernel $\widetilde{r}_{k,\mathrm{Re}(\mu)}$ for the operator $(R_{\mathrm{Re}(\mu)}\eta)^k R_{\mathrm{Re}(\mu)}$ satisfies

$$
|\widetilde{r}_{k,\mathrm{Re}(\mu)}(x,y)| \leqslant c_2 \frac{1}{(2c_1)^k} r_{\mathrm{Re}(\mu)/4}(x-y) \tag{11.6}
$$

for all $x,y \in \mathbb{R}^n$, $x \neq y$, $\mu \in \Sigma_{\mu_0,\vartheta}$, $\eta \in L^\infty(\mathbb{R}^n)$ with $\|\eta\|_{L^\infty} \leqslant \kappa$ and $\operatorname{supp}(\eta) \subset B(0,\tau_2)$ and some $c_2 > 0$, which is possible by Lemma 11.4.

Let $\tau := \min\{\tau_1,\tau_2\}$. Then, for $x,y \in \mathbb{R}^n$, and $\eta \in C^\infty(\mathbb{R}^n)$, with $\operatorname{supp}(\eta) \subset B(0,\tau)$ and $\|\eta\|_{L^\infty} \leqslant \kappa$ one gets for $N, M \in \mathbb{N}$, $N > M$, $\mu \in \Sigma_{\mu_0,\vartheta}$,

$$
\begin{aligned}
\Bigg|\sum_{k=M}^N \big(\underbrace{(r_\mu*\eta)\cdots(r_\mu*\eta)}_{k\text{-times}}\big)r_\mu(x-y)\Bigg| &\leqslant \sum_{k=M}^N c_1^{k+1}|\widetilde{r}_{k,\mathrm{Re}(\mu)}(x,y)| \\
&\leqslant c_2c_1\sum_{k=M}^\infty 2^{-k} r_{\mathrm{Re}(\mu)/4}(x-y) \qquad (11.7) \\
&\leqslant c_2c_1 2^{-M+1} r_{\mathrm{Re}(\mu)/4}(x-y).
\end{aligned}
$$

Thus,

$$\widetilde{r}(x,y) := \sum_{k=0}^{\infty} \big( \underbrace{(r_\mu * \eta) \cdots (r_\mu * \eta)}_{k\text{-times}} \big) r_\mu(x-y), \quad x, y \in \mathbb{R}^n,\ x \neq y,$$

defines a function, which, by the differentiability of $\eta$, coincides with the fundamental solution of $(-\Delta + \eta + \mu)$. For $x, y \in \mathbb{R}^n$, $x \neq y$, $\mu \in \Sigma_{\mu_0,\vartheta}$, one thus gets, using (11.7) for $M = 1$ and $N \to \infty$,

$$\begin{aligned}
|r_{\eta+\mu}(x,y)| &\leqslant |r_{\eta+\mu}(x,y) - r_\mu(x-y)| + |r_\mu(x-y)| \\
&\leqslant c_2 c_1 r_{\mathrm{Re}(\mu)/4}(x-y) + c_1 r_{\mathrm{Re}(\mu)}(x-y) \\
&\leqslant (c_2 c_1 + c_1 2^{\widehat{n}-1}) r_{\mathrm{Re}(\mu)/4}(x-y),
\end{aligned}$$

where, in the last estimate, we used Lemma 5.12, (5.16), with $n = 2\widehat{n} + 1$ for some $\widehat{n} \in \mathbb{N}$. Finally, for $\mu \in \Sigma_{\mu_0,\vartheta}$, and from

$$R_{\eta+\mu} = \sum_{k=0}^{\infty} R_\mu((-\eta)R_\mu)^k = R_\mu - R_\mu \eta R_{\eta+\mu},$$

one reads off, for $x, y \in \mathbb{R}^n$, $x \neq y$,

$$\begin{aligned}
|r_{\eta+\mu}(x,y)| &\leqslant |r_\mu(x-y)| + |r_\mu * \eta r_{\eta+\mu}(x,y)| \\
&\leqslant c_1 r_{\mathrm{Re}(\mu)}(x-y) + c_1 \kappa (c_2 c_1 + c_1 2^{\widehat{n}-1}) r_{\mathrm{Re}(\mu)} * r_{\mathrm{Re}(\mu)/4}(x-y) \\
&\leqslant c_1 \left( 1 + \kappa (c_2 c_1 + c_1 2^{\widehat{n}-1}) \frac{4}{\mu_0} \right) r_{\mathrm{Re}(\mu)}(x-y),
\end{aligned}$$

where we used Proposition 11.6 for $q = r_{\mathrm{Re}(\mu)/4}$ for obtaining the last estimate. □

As a first application of Theorem 11.8, in the spirit of the results derived in Section 5, we can show the following result.

**Corollary 11.9.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, with $n = 2\widehat{n} + 1$ for some $\widehat{n} \in \mathbb{N}$, $\mu_0 > 0$, $\vartheta \in (0, \pi/2)$. Then there exists $\tau > 0$, such that for $m \in \mathbb{N}_{>\widehat{n}}$ there exists $c \geqslant 1$ with the following properties: Given $\Psi_1, \ldots, \Psi_m \in C_b^\infty(\mathbb{R}^n)$, with*

$$|\Psi_j(x)| \leqslant \kappa (1 + |x|)^{-\alpha_j}, \quad x \in \mathbb{R}^n,\ j \in \{1, \ldots, m\},$$

*for some $\alpha_1, \ldots, \alpha_m, \kappa \in [0, \infty)$, then for all $\eta_j \in C_b^\infty(\mathbb{R}^n)$, $\|\eta_j\|_{L^\infty} \leqslant 1$, $j \in \{1, \ldots, m\}$, and $\mathrm{supp}(\eta_j) \subset B(0, \tau)$, the integral kernel $t_\mu$ of $\prod_{j \in \{1,\ldots,m\}} R_{\eta_j+\mu} \Psi_j$ satisfies*

$$|t_\mu(x,x)| \leqslant \kappa^m c (1 + |x|)^{-\sum_{j=1}^m \alpha_j}, \quad x \in \mathbb{R}^n,\ \mu \in \Sigma_{\mu_0,\vartheta}.$$

*Proof.* Choose $\tau > 0$ as the minimum of $\tau$'s according to Theorem 11.8 with $\kappa = 1$ and Lemma 11.2 with $\beta = \frac{1}{2}$. Let $\Psi_1, \ldots, \Psi_m$, $\eta_1, \ldots, \eta_m$, $m$ as in Corollary 11.9, and let $\kappa' > \kappa$. Choose $\widetilde{\Psi}_j \in C_b^\infty(\mathbb{R}^n; [0, \infty))$ with

$$|\Psi_j(x)| \leqslant \widetilde{\Psi}_j(x) \leqslant \kappa' (1 + |x|)^{-\alpha_j}, \quad x \in \mathbb{R}^n,\ j \in \{1, \ldots, m\}.$$

Then, by Theorem 11.8, there exists $c > 0$ with

$$|r_{\eta_j+\mu}(x,y)| \leqslant c r_{\mathrm{Re}(\mu)}(x-y), \quad x, y \in \mathbb{R}^n,\ x \neq y,\ \mu \in \Sigma_{\mu_0,\vartheta}.$$

Hence, for $x \in \mathbb{R}^n$ and $\mu \in \Sigma_{\mu_0,\vartheta}$ one obtains

$$|((r_{\eta_1+\mu} * \Psi_1) \cdots (r_{\eta_m+\mu} * \Psi_m))(x,x)| \leqslant c^m \big( (r_{\mathrm{Re}(\mu)} * \widetilde{\Psi}_1) \cdots (r_{\mathrm{Re}(\mu)} * \widetilde{\Psi}_m) \big)(x,x).$$

Thus, the assertion follows from Lemma 5.14. □

*Remark* 11.10. A result similar to Corollary 11.9 holds if for some index $j \in \{1, \ldots, m\}$, the operator $R_{\eta_j+\mu}$ is replaced by $\partial_\ell R_{\eta_j+\mu}$ for some $\ell \in \{1, \ldots, n\}$. For obtaining such a result, one needs a version of Lemma 5.13 where, in this lemma, the fundamental solution for the Helmholtz equation is replaced by the respective one for $(-\Delta + \eta_j + \mu)u = f$. ⋄

In the rest of this section, we shall establish the remaining estimate needed, to obtain a proof for Remark 11.10. More precisely, we aim for a proof of the following result:

**Theorem 11.11.** *Let $n \in \mathbb{N}_{\geqslant 3}$ odd, for $\mu \in \mathbb{C}_{\mathrm{Re}>0}$, let $q_\mu$ as in Lemma 5.13, $\mu_0 > 0$, $\vartheta \in (0, \pi/2)$, $\kappa > 0$. Then there exists $c \geqslant 1$ and $\tau > 0$ such that for all $j \in \{1, \ldots, n\}$, $\eta \in C_b^\infty(\mathbb{R}^n)$, $\|\eta\|_{L^\infty} \leqslant \kappa$, with $\operatorname{supp}(\eta) \subset B(0, \tau)$, and $\mu \in \Sigma_{\mu_0,\vartheta}$, we have for all $x, y \in \mathbb{R}^n$, $x \neq y$,*

$$|\partial_j(\xi \mapsto r_{\eta+\mu}(\xi, y))(x)| \leqslant c\, q_{\mathrm{Re}(\mu)}(|x-y|)$$

*with $r_{\eta+\mu}$ denoting the integral kernel of $R_{\eta+\mu} = (-\Delta + \eta + \mu)^{-1}$, the latter being given by* (11.1).

The proof of Theorem 11.11 will follow similar ideas as the one for Theorem 11.8. We start with the following result:

**Theorem 11.12.** *Let $n \in \mathbb{N}_{\geqslant 3}$, $k \in \mathbb{N}$, $k < n$, $\mu > 0$. Then the operator*

$$L^2(\mathbb{R}^n) \ni \psi \mapsto \left( x \mapsto \int_{\mathbb{R}^n} \frac{e^{-\mu|x-y|}}{|x-y|^k} \psi(y)\, d^n y \right) \in L^2(\mathbb{R}^n)$$

*is well-defined, bounded, and positive definite.*

*Proof.* The operator is well-defined and bounded by Young's inequality together with the observation that $f \colon x \mapsto e^{-\mu|x|}|x|^{-k}$ is an $L^1(\mathbb{R}^n)$-function. Moreover, for $\varepsilon > 0$ we set

$$\phi_\varepsilon \colon [0, \infty) \to \mathbb{R}, \quad r \mapsto \frac{e^{-\mu r}}{(r+\varepsilon)^k}.$$

Then $\phi_\varepsilon$ is a completely monotone function, since the maps $r \mapsto e^{-\mu r}$ and $r \mapsto (r+\varepsilon)^{-k}$ are completely monotone. Observing that $\phi_\varepsilon(r) \to 0$ as $r \to \infty$ and using the criterion on positive definiteness in [97, Theorem 2] one infers that $\phi_\varepsilon(|\cdot|)*$ is a positive semi-definite operator. Moreover, since $\phi_\varepsilon \to \phi$ in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0$, one gets that $\phi_\varepsilon(|\cdot|)* \to \phi(|\cdot|)*$ in $\mathcal{B}(L^2(\mathbb{R}^n))$ as $\varepsilon \to 0$. Hence, for all $\psi \in L^2(\mathbb{R}^n)$ one infers

$$0 \leqslant \lim_{\varepsilon \to 0} \big(\phi_\varepsilon * \psi, \psi\big)_{L^2(\mathbb{R}^n)} = \big(\phi * \psi, \psi\big)_{L^2(\mathbb{R}^n)}. \qquad \square$$

**Corollary 11.13.** *Let $n \in \mathbb{N}_{\geqslant 3}$, $k \in \mathbb{N}$, $k < n$, $\mu_0, \mu_1 > 0$. Denote $q \colon \mathbb{R}^n \backslash \{0\} \ni x \mapsto e^{-\mu_1|x|}|x|^{-k}$. Then for all $\mu \geqslant \mu_0$,*

$$\mu_0 \int_{\mathbb{R}^n} r_\mu(x - x_1) q(x_1 - x)\, d^n x_1 \leqslant q(x-y), \quad x, y \in \mathbb{R}^n,\ x \neq y, \tag{11.8}$$

*where $r_\mu$ is the integral kernel for $(-\Delta + \mu)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n))$.*

*Proof.* By Theorem 11.12, $q$ satisfies the assumptions in Proposition 11.6, implying inequality (11.8). $\square$

We conclude with the proof of Theorem 11.11, yielding the proof of Remark 11.10.

*Proof of Theorem* 11.11. Choose $\tau > 0$ such that $\|R_\mu \eta\| \leqslant 1/2$ for all $\eta \in L^\infty(\mathbb{R}^n)$, $\|\eta\|_{L^\infty} \leqslant \kappa$, with $\operatorname{supp}(\eta) \subset B(0,\tau)$, and $\mu \in \Sigma_{\mu_0,\vartheta}$, as permitted by Lemma 11.2. Next, let $j \in \{1,\dots,n\}$ and recall

$$\begin{aligned}
\partial_j R_{\eta+\mu} &= \partial_j \sum_{k=0}^{\infty} R_\mu \big((-\eta)R_\mu\big)^k \\
&= \partial_j R_\mu + \partial_j R_\mu (-\eta) R_\mu \sum_{k=1}^{\infty} \big((-\eta)R_\mu\big)^{k-1} \\
&= \partial_j R_\mu + \partial_j R_\mu (-\eta) \sum_{k=0}^{\infty} R_\mu \big((-\eta)R_\mu\big)^k \\
&= \partial_j R_\mu - \partial_j R_\mu (\eta) R_{\eta+\mu}. \qquad (11.9)
\end{aligned}$$

Let $q_\mu$ be as in Lemma 5.13. Upon appealing to Lemma 5.13 (see, in particular, inequalities (5.18) and (5.19)), one is left with estimating the integral kernel associated with the second summand in (11.9), which we denote by $t$. Using Theorem 11.8 and Lemma 5.13, (5.19), there exists $c_1 \geqslant 1$ such that

$$|r_{\eta+\mu}(x,y)| \leqslant c_1 r_{\operatorname{Re}(\mu)/4}(x-y) \text{ and } q_\mu(|x-y|) \leqslant c_1 q_{\operatorname{Re}(\mu)}(|x-y|)$$

for all $\mu \in \Sigma_{\mu_0,\vartheta}$ and $x, y \in \mathbb{R}^n$, $x \neq y$. Thus, for all $x, y \in \mathbb{R}^n$, $x \neq y$, $\mu \in \Sigma_{\mu_0,\vartheta}$, one gets with the help of (11.8) (using that $q_{\operatorname{Re}(\mu)}(|\cdot|)$ is a nonnegative linear combination of functions discussed in Corollary 11.13),

$$\begin{aligned}
|t(x,y)| &= \big|\partial_j r_\mu * (\eta) r_{\eta+\mu}(x,y)\big| \\
&\leqslant c_1 \int_{B(0,\tau)} q_\mu(|x-x_1|)|\eta(x_1)| r_{\operatorname{Re}(\mu)/4}(x_1-y)\, d^n x_1 \\
&\leqslant \|\eta\|_{L^\infty} c_1^2 \int_{\mathbb{R}^n} q_{\operatorname{Re}(\mu)}(|x-x_1|) r_{\operatorname{Re}(\mu)/4}(x_1-y)\, d^n x_1 \\
&\leqslant \|\eta\|_{L^\infty} \frac{4c_1^2}{\mu_0} q_{\operatorname{Re}(\mu)}(|x-y|). \qquad \square
\end{aligned}$$

## 12. The proof of Theorem 10.2: The Smooth Case

In this section, we treat Theorem 10.2 for the particular case of $C^\infty$-potentials[3]. Let $n \in \mathbb{N}_{\geqslant 3}$ odd, $\Phi \in \mathbb{C}_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ for some $d \in \mathbb{N}$. Assume that $\Phi(x) = \Phi(x)^*$ and that for some $c > 0$ and $R > 0$ one has the strict positive definiteness condition $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n\backslash B(0,R)$. With the operator $L = \mathcal{Q} + \Phi$ as in (7.1), we proceed as follows: At first, we show that if $(\mathcal{Q}\Phi)(x) \to 0$, $|x| \to \infty$, then $L$ is a Fredholm operator (Lemma 12.1). Next, if one defines $U \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ to coincide with $\operatorname{sgn}(\Phi)$ on $\mathbb{R}^n\backslash B(0,R)$ as in Lemma 10.4, we show that the operator $\mathcal{Q}+U$ is also a Fredholm operator with the same index (Theorem 12.2). Moreover, in this theorem, we shall also show that changing $U$ to be unitary everywhere but on a small ball around 0 will not change the index. As this ball may be chosen arbitrarily small, we are in the position to proceed with a similar strategy to derive the index as in Section 7 and use the results from Section 11. In that sense, the following may also be considered as a first attempt for a perturbation theory for the generalized Witten index introduced at the end of Theorem 3.4.

We start with the Fredholm property for the operator considered in Theorem 10.2 with smooth potentials (see also Theorem 6.3).

**Lemma 12.1.** *Let $n, d \in \mathbb{N}$, $L = \mathcal{Q} + \Phi$ as in* (7.1)*, with $\Phi \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ and $\Phi(x) = \Phi(x)^*$, $x \in \mathbb{R}^n$. Assume that $C(x) \coloneqq (\mathcal{Q}\Phi)(x) \to 0$ as $|x| \to \infty$ (see also* (6.15)*), and that there exist $c > 0$ and $R > 0$ such that with $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n\backslash B(0,R)$. Then $L$ is a Fredholm operator.*

*Proof.* One recalls from Proposition 6.10 that $L^*L = -\Delta - C + \Phi^2$ and $LL^* = -\Delta + C + \Phi^2$. The latter two operators are $\Delta$-compact perturbations of $-\Delta + \Phi^2$ due to $C(x) \to 0$ as $|x| \to \infty$ and Theorem 6.7. Next, since $-\Delta + \Phi^2 + c^2\chi_{B(0,R)} \geqslant -\Delta + c^2$, the operator $-\Delta+\Phi^2+c^2\chi_{B(0,R)}$ is continuously invertible. But, $-\Delta+\Phi^2+c^2\chi_{B(0,R)}$ is also a $\Delta$-compact perturbation of $-\Delta+\Phi^2$. Thus, by the invariance of the Fredholm property under relatively compact perturbations, one concludes the Fredholm property for $-\Delta + \Phi^2$ and thus the same for $L^*L$ and $LL^*$. $\square$

As a corollary, we obtain the assertion that one might also consider potentials being pointwise unitary outside large balls. In this context, we refer the reader also to the beginning of Section 10. One notes that also Theorem 10.2 hints in the same direction as in the index formula only the sign of the potential occurs.

**Theorem 12.2.** *Let $n, d \in \mathbb{N}$, $L = \mathcal{Q} + \Phi$ as in* (7.1) *with $\Phi \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ with $\Phi(x) = \Phi(x)^*$, $x \in \mathbb{R}^n$. Assume that there exist $c > 0$ and $R > 0$ such that $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n\backslash B(0,R)$ and $(\mathcal{Q}\Phi)(x) \to 0$ as $|x| \to \infty$. If $U \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ pointwise self-adjoint with $\operatorname{sgn}(\Phi(x)) = U(x)$ for all $x \in \mathbb{R}^n$ with $|x| \geqslant R'$ for some $R' \geqslant R$, then $\widetilde{L} \coloneqq \mathcal{Q}+U$ is Fredholm and* $\operatorname{ind}(L) = \operatorname{ind}\big(\widetilde{L}\big)$.

*Proof.* From Lemma 10.4, one gets $(\mathcal{Q}U)(x) = (\mathcal{Q}\operatorname{sgn}(\Phi))(x) \to 0$ as $|x| \to \infty$. Hence, by Lemma 12.1 the operators $L = \mathcal{Q} + \Phi$ and $\widetilde{L} = \mathcal{Q} + U$ are Fredholm (for $\widetilde{L}$ one observes that $U(x)^2 = I_d$ for all $|x| \geqslant R'$). Next, the operator family $[0,1] \ni \lambda \mapsto \mathcal{Q} + (1-\lambda)U + \lambda\min\{c/2,1/2\}U$ defines a homotopy from $\widetilde{L}$ to $\mathcal{Q} + \min\{c/2,1/2\}U$, which is a homotopy of Fredholm operators as $(1-\lambda) + \lambda\min\{c/2,1/2\} = 1 - \lambda(1-\min\{c/2,1/2\}) \geqslant \min\{c/2,1/2\} > 0$ for all $\lambda \in (0,1)$ and hence Lemma 12.1 applies. The rest of the proof is concerned with showing

[3]We note that this section may explain the reasoning underlying the last lines on [22, p. 226].

that $[0,1] \ni \lambda \mapsto \mathcal{Q} + (1-\lambda)\Phi + \lambda \min\{c/2, 1/2\}U$ defines a homotopy of Fredholm operators. Employing Lemma 12.1, it suffices to show that for some $\widetilde{c} > 0$,

$$\big[(1-\lambda)\Phi(x) + \lambda \min\{c/2, 1/2\}U(x)\big]^2 \geqslant \widetilde{c} I_d$$

for all $x \in \mathbb{R}^n \backslash B(0, R')$ and $\lambda \in [0,1]$. By the spectral theorem for symmetric $d \times d$-matrices it suffices to show that for real numbers $\alpha \in \mathbb{R}$ with $\alpha^2 \geqslant c^2$, one has for some $\widetilde{c} > 0$,

$$\big[(1-\lambda)\alpha + \lambda \min\{c/2, 1/2\} \operatorname{sgn}(\alpha)\big]^2 \geqslant \widetilde{c}.$$

But since

$$\big[(1-\lambda)\alpha + \lambda \min\{c/2, 1/2\} \operatorname{sgn}(\alpha)\big]^2 = \big[(1-\lambda)|\alpha| + \lambda \min\{c/2, 1/2\}\big]^2,$$

it remains to observe that

$$(1-\lambda)|\alpha| + \lambda \min\{c/2, 1/2\} \geqslant (1-\lambda)c + \lambda \min\{c/2, 1/2\} \geqslant \min\{c/2, 1/2\}. \quad \square$$

We remark that the assumptions in Theorem 12.2 can be met, using Lemma 10.4. This, and [22] motivates the following notion of "Callias admissibility".

**Definition 12.3.** *Let $n, d \in \mathbb{N}$. We say that a map $\Phi \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ is called* Callias admissible, *if the following conditions (i)–(iii) are satisfied:*

*(i) $\Phi(x) = \Phi(x)^*$, $x \in \mathbb{R}^n$.*

*(ii) There exists $R > 0$ such that $\Phi(x)$ is unitary for all $x \in \mathbb{R}^n \backslash B(0, R)$.*

*(iii) There exists $\varepsilon > 1/2$ such that for all $\alpha \in \mathbb{N}_0^n$, there is $\kappa \geqslant 0$ with*

$$\|\partial^\alpha \Phi(x)\| \leqslant \kappa \begin{cases} (1+|x|)^{-1}, & |\alpha| = 1, \\ (1+|x|)^{-1-\varepsilon}, & |\alpha| \geqslant 2, \end{cases} \quad x \in \mathbb{R}^n.$$

*Remark* 12.4. Let $\Phi \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ be Callias admissible. By Theorem 12.2, for any potential $U \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ coinciding with $\operatorname{sgn}(\Phi)$ on large balls, the operators $L = \mathcal{Q} + \Phi$ and $\widetilde{L} = \mathcal{Q} + U$ are Fredholm with the same index. Moreover, by Theorem 10.3 and the unitarity of $\Phi$ on large balls, one infers that on large balls $U = \Phi$. Next, by Lemma 10.4, we may choose $U$ to be unitary everywhere but on a small ball centered at 0. In addition, we can choose $U$ such that

$$U(x)^2 = u(x) I_d, \quad x \in \mathbb{R}^n,$$

with $u \in C^\infty(\mathbb{R}^n; [0,1])$ and $u = 1$ on $\mathbb{R}^n \backslash B(0, \tau)$ for every chosen $\tau > 0$. For that reason, in order to compute the index for $L = \mathcal{Q} + \Phi$, our main focus only needs to be potentials with the properties of $U$ discussed here, and then one can employ the results of Section 11. $\diamond$

Remark 12.4 leads to the following definition:

**Definition 12.5** ($\tau$-admissibility)**.** *Let $n, d \in \mathbb{R}^n$, $\tau > 0$, $U \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$. We say that $U$ is* admissible on $\mathbb{R}^n \backslash B(0, \tau)$, *in short,* $\tau$-admissible, *if $U$ is Callias admissible and there exists $u \in C^\infty(\mathbb{R}^n; [0,1])$ satisfying*

$$U(x)^2 = u(x) I_d, \quad x \in \mathbb{R}^n, \tag{12.1}$$

*with the property that $u = 1$ on $\mathbb{R}^n \backslash B(0, \tau)$.*

To get a first impression of the difference between the notions of admissibility (see Definition 6.11) and $\tau$-admissibility, we will compute the resolvent differences of the resolvents of $L^*L$ and $LL^*$ in Proposition 12.7, with $L = \mathcal{Q} + U$ given as in (7.1) for some $\tau$-admissible $U$ with $u$ as in (12.4). First, we note that by Proposition 6.10, one has

$$L^*L = -\Delta I_{2^{\widehat{n}}d} + C + uI_{2^{\widehat{n}}d},$$

with $C = (\mathcal{Q}U)$. In Section 7, and in particular in Proposition 7.5, we discussed the resolvent $(L^*L + z)^{-1}$ in terms of $R_{1+z} = (-\Delta I_{2^{\widehat{n}}d} + (1+z))^{-1}$. The latter operator needs to be replaced by the following (see also (11.1))

$$\begin{aligned} R_{u+z} &\coloneqq (-\Delta I_{2^{\widehat{n}}d} + u + z)^{-1} \\ &= \left(-\Delta I_{2^{\widehat{n}}d} + (u-1)I_{2^{\widehat{n}}d} + (z+1)I_{2^{\widehat{n}}d}\right)^{-1} \\ &= \sum_{k=0}^{\infty} (R_{1+z}(u-1)I_{2^{\widehat{n}}d})^k R_{1+z}, \end{aligned} \tag{12.2}$$

provided the latter series converges. As already discussed in Lemma 10.4, this can be ensured if $\tau$ is chosen small enough. Thus, for this pupose, we shall fix the parameters according to the results in Section 11:

**Hypothesis 12.6.** *Let $n = 2\widehat{n} + 1$, $\widehat{n} \in \mathbb{N}_{\geqslant 1}$,*

$$\delta_0 \in (-1, 0), \quad \vartheta \in (0, \pi/2). \tag{12.3}$$

*For $\mu_0 \coloneqq \delta_0 + 1$ let $\tau_{11.2}$ as in Lemma 11.2 for $\beta = 1/2$, $\tau_{11.11}$ as in Theorem 11.11 for $\kappa = 1$, $\tau_{11.8}$ as in Theorem 11.8 for $\kappa = 1$, and $\tau_{11.9}$ as in Corollary 11.9. Define*

$$\tau \coloneqq \min\{\tau_{11.2}, \tau_{11.8}, \tau_{11.9}, \tau_{11.11}\}. \tag{12.4}$$

As mentioned already, for $\tau$-admissible potentials, we shall derive the index theorem similarly to the derivation for admissible potentials. More precisely, at first, we will focus on computing the trace of $\chi_\Lambda B_L(z)$, as in Theorem 7.1. We note that the following parallels the Section 7.

To start, we need to state a result similar to Proposition 7.5. In fact, using the expressions in (7.11) and (7.10), with $R_{1+z}$ replaced by $R_{u+z}$ (see (11.1)), even the proof turns out to be the same.

**Proposition 12.7.** *Assume Hypothesis 12.6, let $z \in \Sigma_{\delta_0,\vartheta}$, and suppose $U \in C_b^\infty(\mathbb{R}^n; \mathbb{C}^{d\times d})$ is $\tau$-admissible (cf. Definition 12.5), with $u$ as in (12.1)). We recall that $L = \mathcal{Q} + U$ as in (7.1), $C = (\mathcal{Q}U)$ in (6.15), and $R_{u+z}$ in (12.2). If, in addition, $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$, then for all $N \in \mathbb{N}$,*

$$\begin{aligned} &(L^*L + z)^{-1} - (LL^* + z)^{-1} \\ &\quad = 2\sum_{k=0}^{N} R_{u+z}\,(CR_{u+z})^{2k+1} + \left((L^*L+z)^{-1} - (LL^*+z)^{-1}\right)(CR_{u+z})^{2N+2} \\ &\quad = 2\sum_{k=0}^{N} R_{u+z}\,(CR_{u+z})^{2k+1} + \left((L^*L+z)^{-1} + (LL^*+z)^{-1}\right)(CR_{u+z})^{2N+3}, \end{aligned}$$

*and*

$$(L^*L + z)^{-1} + (LL^* + z)^{-1}$$

$$= 2\sum_{k=0}^{N} R_{u+z}\,(CR_{u+z})^{2k} + \big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)\,(CR_{u+z})^{2N+2}.$$

Next, we formulate the variant of Lemma 7.7:

**Lemma 12.8.** *Assume Hypothesis* 12.6, *$z \in \Sigma_{\delta_0,\vartheta}$, let $U \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^{d\times d})$ be $\tau$-admissible* (*cf. Definition* 12.5), *with $u$ as in* (12.1). *Let $L = \mathcal{Q} + U$ be given by* (7.1) *and $z \in \Sigma_{\delta_0,\vartheta} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$. We recall $B_L(z)$, $J_L^j(z)$, and $A_L(z)$ given by* (7.2), (7.6), *and* (7.7) (*with $\Phi$ replaced by $U$*), *respectively, as well as $R_{u+z}$ given by* (12.2). *Then the following assertions hold:*

$$\begin{aligned}
2B_L(z) &= \sum_{j=1}^{n}\big[\partial_j, J_L^j(z)\big] + A_L(z),\\
&= z\,\mathrm{tr}_{2^{\hat n}d}\big(2(R_{u+z}C)^n R_{u+z} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)(CR_{u+z})^{n+1}\big),
\end{aligned}$$

*with*

$$\begin{aligned}
J_L^j(z) &= 2\,\mathrm{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}(R_{u+z}C)^{n-2}R_{u+z}\big) + 2\,\mathrm{tr}_{2^{\hat n}d}\big(\gamma_{j,n}U(R_{u+z}C)^{n-1}R_{u+z}\big)\\
&\quad + \mathrm{tr}_{2^{\hat n}d}\big(\gamma_{j,n}\mathcal{Q}\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{u+z})^n\big)\\
&\quad + \mathrm{tr}_{2^{\hat n}d}\big(\gamma_{j,n}U\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{u+z})^n\big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad j \in \{1,\dots,n\},
\end{aligned}$$

*and*

$$\begin{aligned}
A_L(z) &= \mathrm{tr}_{2^{\hat n}d}\big(\big[U, U\big(2(R_{u+z}C)^n R_{u+z} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)\\
&\qquad\qquad \times (CR_{u+z})^{n+1}\big)\big]\big)\\
&\quad - \mathrm{tr}_{2^{\hat n}d}\big(\big[U, \mathcal{Q}\big(2(R_{u+z}C)^{n-1}R_{u+z} + \big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)\\
&\qquad\qquad \times (CR_{u+z})^n\big)\big]\big).
\end{aligned}$$

*Proof.* The proof follows line by line those of Lemma 7.7, observing that $R_{u+z}$ commutes with $\gamma_{j,n}$, $j \in \{1,\dots,n\}$. □

*Remark* 12.9. For even space dimensions $n$ – as in Lemma 7.7 – the corresponding operator $B_L(z)$ also vanishes for all $z \in \varrho(-L^*L) \cap \varrho(-LL^*)$. That is why we will disregard even space dimensions from now on. ◇

The proof of the variant of Theorem 7.8 is slightly more involved:

**Theorem 12.10.** *Assume Hypothesis* 12.6, *$z \in \Sigma_{\delta_0,\vartheta}$. Let $U \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^{d\times d})$ be $\tau$-admissible* (*cf. Definition* 12.5), *with $u$ as in* (12.1). *Let $L = \mathcal{Q} + U$ be given by* (7.1). *Then there exists $\delta_0 \leqslant \delta < 0$, such that for all $z \in \Sigma_{\delta,\vartheta} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$ and $\Lambda > 0$, the operator $\chi_\Lambda B_L(z)$, with $B_L(z)$ given by* (7.2), *is trace class with $z \mapsto \mathrm{tr}(|\chi_\Lambda B_L(z)|)$ bounded on $B(0,|\delta|)\backslash\{0\}$. Moreover, the trace of $\chi_\Lambda B_L(z)$ may be computed as the integral over the diagonal of the corresponding integral kernel.*

*Proof.* It suffices to observe that if $\eta \in L^{n+1}(\mathbb{R}^n)$, one has $R_{u+z}\eta \in \mathcal{B}_{n+1}(L^2(\mathbb{R}^n))$, with

$$\|R_{u+z}\eta\|_{\mathcal{B}_{n+1}} \leqslant 2\|R_{1+z}\eta\|_{\mathcal{B}_{n+1}}.$$

Indeed, from

$$R_{u+z}\eta = \sum_{k=0}^{\infty}(R_{1+z}(u-1))^k R_{1+z}\eta,$$

the ideal property, Hypothesis 12.6, and (11.1), it follows that

$$\|R_{u+z}\eta\|_{\mathcal{B}_{n+1}} = \sum_{k=0}^{\infty} \|(R_{1+z}(u-1))^k\|_{\mathcal{B}_\infty} \|R_{1+z}\eta\|_{\mathcal{B}_{n+1}} \leqslant 2\|R_{1+z}\eta\|_{\mathcal{B}_{n+1}}.$$

The rest of the proof of the trace class property follows literally that of Theorem 7.8. The assertion concerning the computation of the trace rests on Remark 7.9, which applies in this context. $\square$

The variant of Lemma 8.1 with $L = \mathcal{Q} + U$ instead of $L = \mathcal{Q} + \Phi$ for some $\tau$-admissible $U$ need not be stated again as it only contains a statement about the regularity of the integral kernels of $J_L^j(z)$ and $A_L(z)$ (see Lemma 12.8), $j \in \{1, \dots, n\}$. Its proof, however, varies slightly from that of Lemma 8.1 in the sense that $R_{1+z}$ should be replaced by $R_{u+z}$ and $\Phi$ by $U$. In addition, we recall Remark 11.3 $(ii)$ to the effect that the application of $R_{u+z}$ increases weak differentiability by two units.

For the proof of Lemma 8.5, we extensively used that $\mathcal{Q}$ commutes with $R_{1+z}$. However, on notes that $\mathcal{Q}$ does not commute with $R_{u+z}$. In fact, one has

$$[R_{u+z}, \mathcal{Q}] = R_{u+z}\mathcal{Q} - \mathcal{Q}R_{u+z} = R_{u+z}(Qu)R_{u+z},$$

recalling our convention to denote the operator of multipliying with the function $x \mapsto (Qu)(x)$ by $(Qu)$. Due to this lack of commutativity, the proof of the analog to Lemma 8.5 is more involved and expanding the resolvent $R_{u+z}$ in the way done in (12.2), the terms discussed in Lemma 8.5 turn out to be the leading terms in a power series expression:

**Lemma 12.11.** *Assume Hypothesis* 12.6*, let $z \in \Sigma_{\delta_0,\vartheta}$, and suppose that $U \in C_b^\infty(\mathbb{R}^n; \mathbb{C}^{d\times d})$ is $\tau$-admissible (cf. Definition* 12.5*), with $u$ as in* (12.1)*, and $C = (\mathcal{Q}U)$. Let $L = \mathcal{Q} + U$ be given by* (7.1) *and $\chi_\Lambda$ as in* (7.3)*, $\Lambda > 0$. For $z \in \Sigma_{\delta_0,\vartheta}$, $\Lambda > 0$, define*

$$\xi_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\big([\mathcal{Q}, U\,(CR_{u+z})^n]\big)$$

*and*

$$\widetilde{\xi}_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\big([\mathcal{Q}, \mathcal{Q}\,(CR_{u+z})^n]\big).$$

*Then for all $z \in \Sigma_{\delta_0,\vartheta}$, the operators $\xi_\Lambda(z)$, $\widetilde{\xi}_\Lambda(z)$ are trace class and the families*

$$\{z \mapsto \operatorname{tr}_{L^2(\mathbb{R}^n)}(\xi_\Lambda(z))\}_{\Lambda>0} \ \text{ and } \ \big\{z \mapsto \operatorname{tr}_{L^2(\mathbb{R}^n)}\big(\widetilde{\xi}_\Lambda(z)\big)\big\}_{\Lambda>0}$$

*are locally bounded (cf.* (8.1)*).*

*Proof.* As in the proof of Lemma 8.5, we start out with $\xi_\Lambda(z)$ and observe with (11.1),

$$\begin{aligned}
\xi_\Lambda(z) &= \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\big([\mathcal{Q}, U\,(CR_{u+z})^n]\big)\\
&= \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\left(\left[\mathcal{Q}, U\left(C\sum_{k=0}^{\infty}(R_{1+z}(u-1))^k R_{1+z}\right)^n\right]\right)\\
&= \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\Bigg(\Bigg[\mathcal{Q}, U\sum_{k=0}^{\infty}\sum_{\substack{0\leqslant k_1,\dots,k_n\leqslant k\\ k_1+\dots+k_n=k}}\Big(C(R_{1+z}(u-1))^{k_1}R_{1+z}C\\
&\qquad\qquad \times (R_{1+z}(u-1))^{k_2}R_{1+z}\cdots C(R_{1+z}(u-1))^{k_n}R_{1+z}\Big)\Bigg]\Bigg)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{0 \leqslant k_1,\dots,k_n \leqslant k \\ k_1+\dots+k_n=k}} \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d} \Bigg( \Bigg[ \mathcal{Q}, U \\ \times \Big( C(R_{1+z}(u-1))^{k_1} R_{1+z} \cdots C(R_{1+z}(u-1))^{k_n} R_{1+z} \Big) \Bigg] \Bigg).$$

In the expression for $\xi_\Lambda(z)$ just derived, we note that the summand for $k = 0$ has been discussed in Lemma 8.5, so we are left with showing the trace class property for the summands belonging to $k > 0$. Moreover, we need to derive an estimate guaranteeing that the sum in the expression for $\xi_\Lambda(z)$ converges in $\mathcal{B}_1$. Let $k \in \mathbb{N}_{\geqslant 1}$ and $k_1, \dots, k_n \in \mathbb{N}_{\geqslant 0}$ such that $k_1 + \dots + k_n = k$, and consider

$$\begin{aligned} &S_{k_1,\dots,k_n} \\ &\quad \coloneqq \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d} \Big( \Big[ \mathcal{Q}, U \Big( C(R_{1+z}(u-1))^{k_1} R_{1+z} \cdots C(R_{1+z}(u-1))^{k_n} R_{1+z} \Big) \Big] \Big) \\ &\quad = \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d} \Big( \mathcal{Q} U \Big( C(R_{1+z}(u-1))^{k_1} R_{1+z} \cdots C(R_{1+z}(u-1))^{k_n} R_{1+z} \Big) \\ &\qquad - U \Big( C(R_{1+z}(u-1))^{k_1} R_{1+z} \cdots C(R_{1+z}(u-1))^{k_n} R_{1+z} \Big) \mathcal{Q} \Big). \end{aligned} \tag{12.5}$$

Let $j \in \{1, \dots, n\}$ be the smallest index for which $k_j \geqslant 1$. Then the first summand in (12.5) reads

$$\begin{aligned} T &\coloneqq \mathcal{Q} U (C R_{1+z})^{j-1} C (R_{1+z}(u-1))^{k_j} R_{1+z} \cdots C(R_{1+z}(u-1))^{k_n} R_{1+z} \\ &= \mathcal{Q} U (C R_{1+z})^{j-1} C R_{1+z} ((u-1) R_{1+z})^{k_j} \cdots (C R_{1+z})((u-1) R_{1+z})^{k_n} \\ &= \mathcal{Q} U (C R_{1+z})^{j} ((u-1) R_{1+z})^{k_j} \cdots (C R_{1+z})((u-1) R_{1+z})^{k_n}. \end{aligned} \tag{12.6}$$

From

$$\begin{aligned} \mathcal{Q} U \left( C R_{1+z} \right)^j &= U \mathcal{Q} \left( C R_{1+z} \right)^j + [\mathcal{Q}, U] \left( C R_{1+z} \right)^j \\ &= U \Big( \sum_{\ell=1}^{j} (C R_{1+z})^{\ell-1} [\mathcal{Q}, C] R_{1+z} (C R_{1+z})^{j-\ell} + (C R_{1+z})^j \mathcal{Q} \Big) \\ &\quad + [\mathcal{Q}, U] \left( C R_{1+z} \right)^j, \end{aligned}$$

one infers

$$\mathcal{Q} U \left( C R_{1+z} \right)^j \in \mathcal{B}_{(n+1)/j},$$

by Lemma 4.5 and the Hölder-type inequality for the Schatten class operators, Theorem 4.2. On the right-hand side of (12.6), apart from $(C R_{1+z})^j$, there are $n - j$ factors of the form $C R_{1+z} \in \mathcal{B}_{n+1}$. In addition, there is at least one factor $(u-1) R_{1+z} \in \mathcal{B}_{n+1}$, by Lemma 4.5 and the fact that $(u-1) \in L^{n+1}(\mathbb{R}^n)$ (as $(u-1)$ is bounded and compactly supported). Hence, by the trace ideal property and the choice of the parameters as in Hypothesis 12.6, one gets

$$\|T\|_{\mathcal{B}_1} \leqslant \| \mathcal{Q} U \left( C R_{1+z} \right)^j \|_{\mathcal{B}_{n+1)/j}} \| C R_{1+z} \|_{\mathcal{B}_{n+1}}^{n-j} \| (u-1) R_{1+z} \|_{\mathcal{B}_{n+1}} 2^{1-k}.$$

The second term under the trace sign in the expression for $S_{k_1,\dots,k_n}$ (see (12.5)) can be dealt with similarly, so there exists $\kappa > 0$ independently of $\Lambda > 0$, $z \in \Sigma_{\delta_0,\vartheta}$, and $k \in \mathbb{N}$, such that

$$\| S_{k_1,\dots,k_n} \|_{\mathcal{B}_1} \leqslant \kappa 2^{1-k}.$$

Hence, for all $\Lambda > 0$ and $z \in \Sigma_{\delta_0,\vartheta}$ one gets

$$\begin{aligned} \|\xi_\Lambda(z)\|_{\mathcal{B}_1} &\leqslant \|\psi_\Lambda(z)\|_{\mathcal{B}_1} + \sum_{k=1}^{\infty} \sum_{\substack{0\leqslant k_1,\dots,k_n\leqslant k \\ k_1+\dots+k_n=k}} \|S_{k_1,\dots,k_n}\|_{\mathcal{B}_1} \\ &\leqslant \|\psi_\Lambda(z)\|_{\mathcal{B}_1} + \sum_{k=1}^{\infty} \kappa(k+1)^n 2^{1-k}, \end{aligned} \tag{12.7}$$

where $\psi_\Lambda(z)$ is defined in Lemma 8.5. Inequality (12.7) yields the assertion for $\xi_\Lambda$.

A similar reasoning – as in Lemma 8.5 for $\widetilde{\psi}_\Lambda(z)$ – applies to $\widetilde{\xi}_\Lambda(z)$. □

Next, we turn to the proof of a modified version of Lemma 8.6:

**Lemma 12.12.** *Assume Hypothesis* 12.6. *Let $z \in \Sigma_{\delta_0,\vartheta}$, and assume that $U \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ is $\tau$-admissible* (*cf. Definition* 12.5), *with $u$ as in* (12.1), $C = (\mathcal{Q}U)$. *Let $L = \mathcal{Q} + U$ be given by* (7.1) *and $\chi_\Lambda$ as in* (7.3), $\Lambda > 0$. *For $z \in \Sigma_{\delta_0,\vartheta}$, $\Lambda > 0$, define*

$$\zeta_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\Big(\big[\mathcal{Q}, U\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{u+z})^{n+1}\,\big]\Big)$$

*and*

$$\widetilde{\zeta}_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\Big(\big[\mathcal{Q}, \big(\mathcal{Q}\big(\,(L^*L+z)^{-1} - (LL^*+z)^{-1}\,\big)\,(CR_{u+z})^{n+1}\,\big)\big]\Big).$$

*Then for all $z \in \Sigma_{\delta_0,\vartheta} \cap \varrho(-L^*L) \cap \varrho(-LL^*)$, the operators $\zeta_\Lambda(z)$, $\widetilde{\zeta}_\Lambda(z)$ are trace class and there exists $\delta \in (\delta_0, 0)$ such that the families*

$$\{\Sigma_{\delta,\vartheta} \ni z \mapsto z \operatorname{tr}_{L^2(\mathbb{R}^n)}(\zeta_\Lambda(z))\}_{\Lambda>0} \ \text{ and } \ \big\{\Sigma_{\delta,\vartheta} \ni z \mapsto z \operatorname{tr}_{L^2(\mathbb{R}^n)}\big(\widetilde{\zeta}_\Lambda(z)\big)\big\}_{\Lambda>0}$$

*are locally bounded* (*cf.* (8.1)).

*Proof.* On can follow the proof of Lemma 8.6 line by line upon replacing $\Phi$ by $U$ and $R_{1+z}$ by $R_{u+z}$. (We recall $CR_{u+z} \in \mathcal{B}_{n+1}$ with $\|CR_{u+z}\|_{\mathcal{B}_{n+1}} \leqslant 2\|CR_{1+z}\|_{\mathcal{B}_{n+1}}$). □

As in the derivation of Theorem 7.1 we summarize the results obtained for local boundedness in a theorem (cf. Theorem 8.7):

**Theorem 12.13.** *Assume Hypothesis* 12.6, $z \in \Sigma_{\delta_0,\vartheta}$. *Let $U \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ be $\tau$-admissible* (*cf. Definition* 12.5), *with $u$ as in* (12.1), $C = (\mathcal{Q}U)$. *Let $L = \mathcal{Q}+U$ be given by* (7.1) *and $\chi_\Lambda$ as in* (7.3), $\Lambda > 0$. *Define for $z \in \Sigma_{\delta_0,\vartheta} \cap \varrho(-LL^*) \cap \varrho(-L^*L)$,*

$$\iota_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\Big(\big[\mathcal{Q}, U\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{u+z})^n\big]\Big),$$

*and*

$$\widetilde{\iota}_\Lambda(z) := \chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}\Big(\big[\mathcal{Q}, \mathcal{Q}\big((L^*L+z)^{-1} + (LL^*+z)^{-1}\big)(CR_{u+z})^n\big]\Big).$$

*Then for all $z \in \Sigma_{\delta_0,\vartheta} \cap \varrho(-LL^*) \cap \varrho(-L^*L)$, the operators $\iota_\Lambda(z)$, $\widetilde{\iota}_\Lambda(z)$ are trace class and there exists $\delta \in (\delta_0, 0)$ such that the families*

$$\{\Sigma_{\delta,\vartheta} \ni z \mapsto z \operatorname{tr}_{L^2(\mathbb{R}^n)}(\iota_\Lambda(z))\}_{\Lambda>0} \ \text{ and } \ \big\{\Sigma_{\delta,\vartheta} \ni z \mapsto z \operatorname{tr}_{L^2(\mathbb{R}^n)}\big(\widetilde{\iota}_\Lambda(z)\big)\big\}_{\Lambda>0}$$

*are locally bounded* (*cf.* (8.1)).

*Proof.* As in the proof for Theorem 12.13, it suffices to realize that $\iota_\Lambda(z) = 2\xi_\Lambda(z) + \zeta_\Lambda(z)$ and $\widetilde{\iota}_\Lambda(z) = 2\widetilde{\xi}_\Lambda(z) + \widetilde{\zeta}_\Lambda(z)$ with the functions introduced in Lemmas 12.11 and 12.12. Thus, the assertion follows from the Lemmas 12.11 and 12.12. □

The proof of the result analogous to Lemma 8.10 needs some modifications. In particular, one should pay particular attention to the assertion concerning $h_{1,j}$: In Lemma 8.10 we proved that $h_{1,j}$ vanishes on the diagonal. Here, we are only able to give an estimate.

**Lemma 12.14.** *Assume Hypothesis* 12.6. *Let $z \in \Sigma_{\delta_0,\vartheta}$, and assume that $U \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ is $\tau$-admissible (cf. Definition* 12.5*), with $u$ as in* (12.1)*, $C = (\mathcal{Q}U)$. Let $L = \mathcal{Q} + U$ be given by* (7.1)*, $R_{u+z}$ as in* (11.1) *as well as $\mathcal{Q}$, and $\gamma_{j,n}$, $j \in \{1,\dots,n\}$, given by* (6.3)*, and as in Remark* 6.1*, respectively. Then for $n \geqslant 3$, the integral kernel $h_{2,j}(z)$ of*

$$2\operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}U\left(R_{u+z}C\right)^{n-1}R_{u+z}\right)$$

*satisfies,*

$$h_{2,j}(z)(x,x) = h_{3,j}(z)(x,x) + g_{0,j}(z)(x,x),$$

*where $h_{3,j}(z)$ is the integral kernel of $2\operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{u+z}^n\right)$ and $g_{0,j}(z)$ satisfies*

$$\sup_{z\in\Sigma_{\delta_0,\vartheta}} |g_{0,j}(z)(x,x)| \leqslant \kappa(1+|x|)^{1-n-\varepsilon}.$$

*for all $x \in \mathbb{R}^n$ and some $\kappa > 0$.*
*In addition, if $n \geqslant 5$ and $z \in \mathbb{R}$, then the integral kernel $h_{1,j}(z)$ of*

$$\operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{u+z}C\right)^{n-2}R_{u+z}\right)$$

*satisfies*

$$\sup_{z\in\Sigma_{\delta_0,\vartheta}} |h_{1,j}(z)(x,x)| \leqslant \kappa(1+|x|)^{-n}.$$

*Proof.* We start with $h_{1,j}(z)$. Using the Neumann series expression in (11.1), one computes,

$$\begin{aligned}
H_{1,j}(z) &:= \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{u+z}C\right)^{n-2}R_{u+z}\right)\\
&= \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(\sum_{k=0}^\infty (R_{1+z}(u-1))^k R_{1+z}C\right)^{n-2}\sum_{k=0}^\infty (R_{1+z}(u-1))^k R_{1+z}\right)\\
&= \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}C\right)^{n-2}R_{1+z}\right)\\
&\quad + \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(\sum_{k=1}^\infty (R_{1+z}(u-1))^k R_{1+z}C\right)\right.\\
&\qquad \times\left.\left(\sum_{k=0}^\infty (R_{1+z}(u-1))^k R_{1+z}C\right)^{n-3}\times\sum_{k=0}^\infty (R_{1+z}(u-1))^k R_{1+z}\right)\\
&\quad + \cdots + \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(\sum_{k=0}^\infty (R_{1+z}(u-1))^k R_{1+z}C\right)^{n-2}\right.\\
&\qquad\qquad \times\left.\sum_{k=1}^\infty (R_{1+z}(u-1))^k R_{1+z}\right)\\
&= \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}(R_{1+z}C)^{n-2}R_{1+z}\right)\\
&\quad + \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}(R_{1+z}(u-1))(R_{u+z}C)^{n-2}R_{u+z}\right)\\
&\quad + \cdots + \operatorname{tr}_{2^{\widehat{n}}d}\left(\gamma_{j,n}\mathcal{Q}(R_{u+z}C)^{n-2}(R_{1+z}(u-1))R_{u+z}\right).
\end{aligned}$$

By Lemma 8.10, the diagonal of the integral kernel associated with

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{1+z}C\right)^{n-2}R_{1+z}\right)$$

vanishes. Thus, it remains to address the asymptotics of the diagonal of the integral kernel associated with

$$\begin{aligned}&\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}(R_{1+z}(u-1))(R_{u+z}C)^{n-2}R_{u+z}\right)\\&\quad+\cdots+\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}(R_{u+z}C)^{n-2}(R_{1+z}(u-1))R_{u+z}\right).\end{aligned}$$

One observes that the function $(u-1)$ vanishes outside $B(0,\tau)\subset\mathbb{R}^n$ (we recall Hypothesis 12.6). Being bounded by 1, it particularly satisfies the estimate

$$|(u-1)(x)|\leqslant(1+\tau)^n(1+|x|)^{-n},\quad x\in\mathbb{R}^n.$$

Realizing that the function $C$ is bounded, the assertion for $h_{1,j}(z)$ follows from Remark 11.10.

The assertion about $h_{2,j}$ can be shown with Remark 5.18 (replacing the operators $R_\mu$ in that remark by $R_{u+z}$ and using that the integral kernel of $R_{u+z}$ can be estimated by the respective one for $R_{1+\operatorname{Re}(z)}$, see Theorem 11.8) and the asymptotic conditions imposed on $U$ (see Definition 12.5). $\square$

The analog of Theorem 8.11, stated below, is now shown in the same way, employing Theorems 11.11 and 11.8:

**Theorem 12.15.** *Assume Hypothesis* 12.6*,* $z\in\Sigma_{\delta_0,\vartheta}$*. Let* $U\in C_b^\infty\left(\mathbb{R}^n;\mathbb{C}^{d\times d}\right)$ *be* $\tau$*-admissible* (*cf. Definition* 12.5)*, with* $u$ *as in* (12.1)*,* $C=(\mathcal{Q}U)$*. Let* $L=\mathcal{Q}+U$ *be given by* (7.1)*,* $R_{u+z}$ *as in* (11.1) *as well as* $\mathcal{Q}$*, and* $\gamma_{j,n}$*,* $j\in\{1,\dots,n\}$*, given by* (6.3)*, and as in Remark* 6.1*, respectively. Then there exists* $z_0>0$*, such that for all* $z\in\mathbb{C}$ *with* $\operatorname{Re}(z)>z_0$*, the integral kernels* $g_1$ *and* $g_2$ *of the operators*

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}U\left(\,(L^*L+z)^{-1}+(LL^*+z)^{-1}\,\right)(CR_{u+z})^n\,\right)$$

*and*

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}Q\left((L^*L+z)^{-1}+(LL^*+z)^{-1}\right)(CR_{u+z})^n\,\right),$$

*respectively, satisfy for some* $\kappa>0$*,*

$$(|g_1(x,x)|+|g_2(x,x)|)\leqslant\kappa(1+|x|)^{-n},\quad x\in\mathbb{R}^n.$$

The next result, the analog of Corollary 8.12, is slightly different compared to the previous analogs since the operator $\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{1+z}^n\right)$ is *not* replaced by $\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{u+z}^n\right)$. Indeed, Corollary 8.12 was used to show that the only important term for the computation for the index is given by $\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{1+z}^n\right)$, for which we computed the integral over the diagonal of the corresponding integral kernel in Proposition 8.13, eventually yielding the formula for the index. Since the asserted formulas for admissible and $\tau$-admissible potentials are the same, we need to have a result to the effect that the integral of over the diagonal of the integral kernels of the operators $\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{u+z}^n\right)$ and $\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{1+z}^n\right)$ should lead to the same results. In fact, this is part of the proof of the following result:

**Corollary 12.16.** *Assume Hypothesis* 12.6*,* $z\in\Sigma_{\delta_0,\vartheta}$*. Let* $U\in C_b^\infty\left(\mathbb{R}^n;\mathbb{C}^{d\times d}\right)$ *be* $\tau$*-admissible* (*cf. Definition* 12.5)*, with* $u$ *as in* (12.1)*,* $C=(\mathcal{Q}U)$*. Let* $L=\mathcal{Q}+U$ *be given by* (7.1)*,* $R_{u+z}$ *as in* (11.1) *as well as* $\mathcal{Q}$*, and* $\gamma_{j,n}$*,* $j\in\{1,\dots,n\}$*, given by* (6.3)*, and as in Remark* 6.1*, respectively.*

(*i*) *Let* $n\in\mathbb{N}_{\geqslant5}$*,* $j\in\{1,\dots,n\}$*. Then there exists* $z_0>0$*, such that if* $z\in\mathbb{C}$*,*

$\operatorname{Re}(z) > z_0$, *and* $h$ *and* $g$ *denote the integral kernel of* $2\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{1+z}^n\right)$ *and* $J_L^j(z)$, *respectively, then for some* $\kappa > 0$,

$$|h(x,x) - g(x,x)| \leqslant \kappa(1+|x|)^{1-n-\varepsilon}, \quad x \in \mathbb{R}^n.$$

$(ii)$ *The assertion of part* $(i)$ *also holds for* $n = 3$, *if, in the above statement,* $J_L^j(z)$ *is replaced by* $J_L^j(z) - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})$.

*Proof.* One recalls from Lemma 12.8,

$$\begin{aligned} J_L^j(z) &= 2\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\left(R_{u+z}C\right)^{n-2}R_{u+z}\right) + 2\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}U\left(R_{u+z}C\right)^{n-1}R_{u+z}\right) \\ &\quad + \operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}\mathcal{Q}\big(\left(L^*L+z\right)^{-1} + \left(LL^*+z\right)^{-1}\big)\left(CR_{u+z}\right)^n\right) \\ &\quad + \operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}U\big(\left(L^*L+z\right)^{-1} + \left(LL^*+z\right)^{-1}\big)\left(CR_{u+z}\right)^n\right). \end{aligned}$$

With the help of Theorem 12.15 one deduces that the integral kernels of the last two terms may be estimated by $\kappa(1+|x|)^{-n}$ on the diagonal. The integral kernel of the first term is also bounded by $\kappa'(1+|x|)^{-n}$ for a suitable $\kappa'$ by Lemma 12.14. Hence, it remains to inspect the second term on the right-hand side. The assertion follows from Lemma 12.14 once we establish estimates for the respective integral kernels of the differences

$$\operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{u+z}^n\right) - \operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{1+z}^n\right) \tag{12.8}$$

and

$$\operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{u+z}CR_{u+z}\right) - \operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}\right). \tag{12.9}$$

In this context we will use the equation

$$\begin{aligned} R_{u+z} &= \sum_{k=0}^{\infty} R_{1+z}((u-1)R_{1+z})^k \\ &= R_{1+z} + \sum_{k=1}^{\infty} R_{1+z}((u-1)R_{1+z})^k \\ &= R_{1+z} + R_{1+z}(u-1)R_{u+z}. \end{aligned}$$

Thus, (12.9) and (12.8) read

$$\begin{aligned} &\operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}(u-1)R_{u+z}CR_{1+z}\right) + \operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}(u-1)R_{u+z}\right) + \\ &\quad + \operatorname{tr}_{2d}\left(\gamma_{j,3}\mathcal{Q}R_{1+z}(u-1)R_{u+z}CR_{1+z}(u-1)R_{u+z}\right) \end{aligned} \tag{12.10}$$

and

$$\sum_{k=1}^{n} \operatorname{tr}_{2^{\hat{n}}d}\left(\gamma_{j,n}UC^{n-1}R_{u+z}^{k-1}R_{1+z}(u-1)R_{u+z}R_{u+z}^{n-k}\right), \tag{12.11}$$

respectively. In each summand of (12.10) and (12.11) there is one term $(u-1)$ which is compactly supported and thus clearly satisfies for some $\kappa' > 0$, $|(u-1)(x)| \leqslant \kappa'(1+|x|)^{-n}$, $x \in \mathbb{R}^n$. Hence, the assertion on the asymptotics of the integral kernels associated with the operators in (12.9) and (12.8) follows from Corollary 11.9. □

We are now ready to prove Theorem 10.2 for $R > 0$ and smooth potentials $\Phi$.

**Theorem 12.17** (Theorem 10.2 for $R > 0$, smooth case)**.** *Let* $n, d \in \mathbb{N}$, $n \geqslant 3$ *odd, and* $\Phi \in C_b^\infty\left(\mathbb{R}^n, \mathbb{C}^{d\times d}\right)$ *satisfy the following assumptions:*

$(i)$ $\Phi(x) = \Phi(x)^*$, $x \in \mathbb{R}^n$.

*(ii) There exist $c > 0$ and $R > 0$ such that $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n \backslash B(0,R)$.*

*(iii) There exists $\varepsilon > 1/2$ such that for all $\alpha \in \mathbb{N}_0^n$ there is $\kappa > 0$ with*

$$\|\partial^\alpha \Phi(x)\| \leqslant \kappa \begin{cases} (1+|x|)^{-1}, & |\alpha| = 1, \\ (1+|x|)^{-1-\varepsilon}, & |\alpha| \geqslant 2, \end{cases} \quad x \in \mathbb{R}^n.$$

*Let $\widetilde{L} = \mathcal{Q} + \Phi$ as in (7.1), $\delta_0 \in (-1,0), \vartheta \in (0,\pi/2)$. Then there exists $\tau > 0$ such that for all $\tau$-admissible potentials $U$ with $U = \operatorname{sgn}(\Phi)$ on sufficiently large balls, and with $L := \mathcal{Q} + U$, the following assertions $(\alpha)$–$(\delta)$ hold:*

($\alpha$) *There exists $\delta_0 \leqslant \delta < 0$ and $0 < \vartheta \leqslant \vartheta_0$ such that for all $\Lambda > 0$ the family*

$$\Sigma_{\delta,\vartheta} \ni z \mapsto z\chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}((L^*L+z)^{-1} - (LL^*+z)^{-1}) \in \mathcal{B}_1(L^2(\mathbb{R}^n)) \quad (12.12)$$

*is analytic.*

($\beta$) *The family $\{f_\Lambda\}_{\Lambda>0}$ of holomorphic functions*

$$f_\Lambda \colon \Sigma_{\delta,\vartheta} \ni z \mapsto \operatorname{tr}\left(z\chi_\Lambda \operatorname{tr}_{2^{\hat{n}}d}((L^*L+z)^{-1} - (LL^*+z)^{-1})\right) \quad (12.13)$$

*is locally bounded (see (8.1)).*

($\gamma$) *The limit $f := \lim_{\Lambda\to\infty} f_\Lambda$ exists in the compact open topology and satisfies for all $z \in \Sigma_{\delta,\vartheta}$,*

$$\begin{aligned} f(z) = c_n(1+z)^{-n/2} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}} \\ \times \int_{\Lambda S^{n-1}} \operatorname{tr}(U(x)(\partial_{i_1}U(x)\dots(\partial_{i_{n-1}}U)(x))x_j \, d^{n-1}\sigma(x), \end{aligned} \quad (12.14)$$

*where*

$$c_n := \frac{1}{2}\left(\frac{i}{8\pi}\right)^{(n-1)/2} \frac{1}{[(n-1)/2]!}.$$

($\delta$) *the operators $\widetilde{L}$ and $L$ are Fredholm operators and*

$$\begin{aligned} \operatorname{ind}\left(\widetilde{L}\right) &= \operatorname{ind}(L) = f(0) \\ &= c_n \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}} \\ &\quad \times \int_{\Lambda S^{n-1}} \operatorname{tr}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x))x_j \, d^{n-1}\sigma(x). \end{aligned} \quad (12.15)$$

As mentioned earlier in connection with the proof of Theorem 12.17, we will follow the analogous reasoning used for the proof for Theorem 7.1. So for the proof of Theorem 12.17 we now need to replace the statements Theorem 7.8, Lemma 7.7, Lemma 8.10, Theorem 8.7 and Corollary 8.12 by the respective results Theorem 12.10, Lemma 12.8, Lemma 12.14, Theorem 12.13 and Corollary 12.16 obtained in this section. Since large parts of the proof would just be a repetition of arguments used in the proof of Theorem 7.1, we will not give a detailed proof for the case $n \geqslant 5$. However, in Section 8, we only sketched how the result for $n = 3$ comes about. As this case is notationally less messy, we will now give the full proof for the case $n = 3$. As in Section 9, the core idea is to regularize the expressions involved

by multiplying $B_L(z)$ with $(1-\mu\Delta)^{-1}$, $\mu>0$, from either side, which results in

$$B_{L,\mu}(z) = (1-\mu\Delta)^{-1}\, B_L(z)\, (1-\mu\Delta)^{-1},$$

(compare with (9.1)), and similarly for $J^j_{L,\mu}(z)$ and $A_{L,\mu}(z)$, recalling (9.2) and (9.3), respectively.

*Proof of Theorem* 12.17, $n=3$. Part ($\alpha$): This follows from Theorem 12.10.

Part ($\beta$): Again by Theorem 12.10, the expression $\operatorname{tr}(\chi_\Lambda B_L(z))$, with $B_L(z)$ as given in (3.2), can be computed as the integral over the diagonal of its integral kernel. Next, we denote by $\mathbb{A}$ and $\mathbb{J}$ the integral kernels for the operators $A_L(z)$ and $2B_L(z)-A_L(z)$, respectively, and correspondingly $\mathbb{A}_\mu$ and $\mathbb{J}_\mu$ for $A_{L,\mu}(z)$ and $2B_{L,\mu}(z)-A_{L,\mu}(z)$, $\mu>0$. Hence, Proposition 5.5 applied to $\mathbb{A}$ yields,

$$\begin{aligned} 2f_\Lambda(z) &= 2\operatorname{tr}(\chi_\Lambda B_L(z)) \\ &= \int_{B(0,\Lambda)} \mathbb{A}(x,x)+\mathbb{J}(x,x)\, d^3x = \int_{B(0,\Lambda)} \mathbb{J}(x,x)\, d^3x \\ &= \lim_{\mu\to 0}\int_{B(0,\Lambda)} \mathbb{J}_\mu(x,x)\, d^3x, \end{aligned}$$

where in the last equality we used the continuity the integral kernels of $2B_L(z)$ and $A_L(z)$, as well as Lemma 8.9.

Next, appealing to the analog result of Lemma 9.2 for $\Phi$ being replaced by the $\tau$-admissible potential $U$, one arrives at

$$\begin{aligned} 2f_\Lambda(z) &= \lim_{\mu\to 0}\int_{B(0,\Lambda)} \mathbb{J}_\mu(x,x)\, d^3x \\ &= \lim_{\mu\to 0}\int_{B(0,\Lambda)} \left\langle \delta_{\{x\}}, \sum_{j=1}^{3}[\partial_j, J^j_{L,\mu}(z)]\delta_{\{x\}}\right\rangle d^3x. \end{aligned}$$

Denote

$$\mathbb{K}_{L,\mu} := \left\{x\mapsto g^j_{L,\mu}(z)(x,x)\right\}_{j\in\{1,2,3\}},$$

where $g^j_{L,\mu}(z)$ is the integral kernel of $J^j_{L,\mu}(z)$, $j\in\{1,2,3\}$, and $\mathbb{K}_{L,z}$ that for

$$\left\{J^j_L(z) - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})\right\}_{j\in\{1,2,3\}}.$$

Invoking Lemmas 9.4 and 8.9, and hence the fact that $\{x\mapsto \mathbb{K}_{L,\mu}(x,x)\}_{\mu>0}$ is locally bounded, one obtains

$$\begin{aligned} \lim_{\mu\to 0}\int_{B(0,\Lambda)} \mathbb{J}_{L,\mu}(x,x)\, d^3x &= \lim_{\mu\to 0}\int_{\Lambda S^2}\left(\mathbb{K}_{L,\mu}(x,x),\frac{x}{\Lambda}\right) d^2\sigma(x) \\ &= \int_{\Lambda S^2}\lim_{\mu\to 0}\left(\mathbb{K}_{L,\mu}(x,x),\frac{x}{\Lambda}\right) d^2\sigma(x) \\ &= \int_{\Lambda S^2}\left(\mathbb{K}_{L,z}(x,x),\frac{x}{\Lambda}\right) d^2\sigma(x). \end{aligned}$$

Hence, one arrives at

$$\begin{aligned} 2f_\Lambda(z) &= \int_{\Lambda S^2}\left(\mathbb{K}_{L,z}(x),\frac{x}{\Lambda}\right) d^2\sigma(x) \qquad (12.16)\\ &= \int_{B(0,\Lambda)} \left\langle \delta_{\{x\}}, \sum_{j=1}^{3}[\partial_j, \widetilde{J}^j_L(z)]\delta_{\{x\}}\right\rangle d^3x, \end{aligned}$$

with
$$\widetilde{J}_L^j(z) \coloneqq J_L^j(z) - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}).$$
For proving that $\{f_\Lambda\}_{\Lambda>0}$ is locally bounded, we recall from Lemma 12.8 that
$$\begin{aligned}\widetilde{J}_L^j(z) = {}& 2\operatorname{tr}_{2d}\big(\gamma_{j,3}\mathcal{Q}(R_{u+z}C)R_{u+z}\big) - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z}) \\ &+ 2\operatorname{tr}_{2d}\big(\gamma_{j,3}U(R_{u+z}C)^2R_{u+z}\big) \\ &+ \operatorname{tr}_{2d}\big(\gamma_{j,3}\mathcal{Q}\big((L^*L+z)^{-1}+(LL^*+z)^{-1}\big)(CR_{u+z})^3\big) \\ &+ \operatorname{tr}_{2d}\big(\gamma_{j,3}U\big((L^*L+z)^{-1}+(LL^*+z)^{-1}\big)(CR_{u+z})^3\big),\end{aligned}$$
and, thus,
$$\begin{aligned}&\sum_{j=1}^3 \big[\partial_j, \widetilde{J}_L^j(z)\big] \\ &\quad = \sum_{j=1}^3 \big[\partial_j, \big(2\operatorname{tr}_{2d}\big(\gamma_{j,3}\mathcal{Q}(R_{u+z}C)R_{u+z}\big) - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})\big)\big] \\ &\qquad + \sum_{j=1}^3 \big[\partial_j, 2\operatorname{tr}_{2d}\big(\gamma_{j,3}U(R_{u+z}C)^2R_{u+z}\big)\big] \\ &\qquad + \operatorname{tr}_{2d}\big(\big[\mathcal{Q}, U\big((L^*L+z)^{-1}+(LL^*+z)^{-1}\big)(CR_{u+z})^3\big]\big) \\ &\qquad + \operatorname{tr}_{2d}\big(\big[\mathcal{Q}, \big(\mathcal{Q}\big((L^*L+z)^{-1}+(LL^*+z)^{-1}\big)(CR_{u+z})^3\big)\big]\big).\end{aligned}$$
Hence,
$$\begin{aligned}2f_\Lambda(z) = {}& \int_{B(0,\Lambda)} \bigg\langle \delta_{\{x\}}, \sum_{j=1}^3 \bigg[\partial_j, \big(2\operatorname{tr}_{2d}\big(\gamma_{j,3}\mathcal{Q}(R_{u+z}C)R_{u+z}\big) \\ &\qquad - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})\big)\bigg]\delta_{\{x\}}\bigg\rangle\, d^3x \\ &+ \int_{B(0,\Lambda)} \bigg\langle \delta_{\{x\}}, \sum_{j=1}^3 \big[\partial_j, 2\operatorname{tr}_{2d}\big(\gamma_{j,3}U(R_{u+z}C)^2R_{u+z}\big)\big]\delta_{\{x\}}\bigg\rangle\, d^3x \\ &+ \operatorname{tr}(\iota_\Lambda(z)) + \operatorname{tr}(\widetilde{\iota}_\Lambda(z)),\end{aligned}$$
where $\iota_\Lambda(z)$ and $\widetilde{\iota}_\Lambda(z)$ are defined in Theorem 12.13. With the help of part $(\alpha)$, Theorem 12.13 and Lemma 8.3, to prove part $(\beta)$, it suffices to prove the local boundedness of
$$\begin{aligned}z \mapsto \int_{B(0,\Lambda)} \bigg\langle \delta_{\{x\}}, \sum_{j=1}^3 \big[\partial_j, \big(2\operatorname{tr}_{2d}\big(\gamma_{j,3}\mathcal{Q}(R_{u+z}C)R_{u+z}\big) \\ - 2\operatorname{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})\big)\big]\delta_{\{x\}}\bigg\rangle\, d^3x\end{aligned}$$
and
$$z \mapsto \int_{B(0,\Lambda)} \bigg\langle \delta_{\{x\}}, \sum_{j=1}^3 \big[\partial_j, 2\operatorname{tr}_{2d}\big(\gamma_{j,3}U(R_{u+z}C)^2R_{u+z}\big)\big]\delta_{\{x\}}\bigg\rangle\, d^3x,$$
both considered as families of functions indexed by $\Lambda > 0$. Appealing to Gauss' divergence theorem, it suffices to show that the integral kernels associated with the

operators

$$\begin{aligned}
&\mathcal{Q}R_{u+z}CR_{u+z} - \mathcal{Q}R_{1+z}CR_{1+z} \\
&\quad = \mathcal{Q}R_{1+z}(u-1)R_{u+z}CR_{u+z} + \mathcal{Q}R_{u+z}CR_{1+z}(u-1)R_{u+z} \\
&\qquad + \mathcal{Q}R_{1+z}(u-1)R_{u+z}CR_{1+z}(u-1)R_{u+z}
\end{aligned}$$

and

$$U(R_{u+z}C)^2R_{u+z}$$

can be estimated on the diagonal by $\kappa'(1+|x|)^{-2}$ for some $\kappa' > 0$ for sufficiently large $|x|$. However, this is a consequence of Corollary 11.9, proving part $(\beta)$.

Part $(\gamma)$: By Montel's Theorem, there exists a sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ of positive reals tending to infinity such that $f := \lim_{k\to\infty} f_{\Lambda_k}$ exists in the compact open topology. From (12.16), one recalls

$$2f_\Lambda(z) = \int_{\Lambda S^2} \left(\mathbb{K}_{L,z}(x,x), \frac{x}{\Lambda}\right) d^2\sigma(x),$$

with $\mathbb{K}_{L,z}$ denoting the integral kernel of

$$\left\{J_L^j(z) - 2\,\mathrm{tr}_{2d}(\gamma_{j,3}\mathcal{Q}R_{1+z}CR_{1+z})\right\}_{j\in\{1,2,3\}}.$$

Next, we choose $z_0 > 0$ as in Corollary 12.16 $(ii)$ and let $z \in \Sigma_{z_0,\vartheta}$. With $h_j$, the integral kernel of $2\mathrm{tr}_{2d}\left(\gamma_{j,3}\Psi C^2 R_{1+z}^3\right)$, we define $\mathbb{H}_z := \left(x \mapsto h_j(x,x)\right)_{j\in\{1,2,3\}}$. Due to Corollary 12.16 $(ii)$ one can find $\kappa > 0$ such that for $k \in \mathbb{N}$,

$$\begin{aligned}
&\left|\int_{\Lambda_k S^2} \left((\mathbb{K}_{J,z} - \mathbb{H}_z)(x), \frac{x}{\Lambda_k}\right)_{\mathbb{R}^n} d^2\sigma(x)\right| \\
&\quad \leqslant \int_{\Lambda_k S^2} \|(\mathbb{K}_{J,z} - \mathbb{H}_z)(x)\|_{\mathbb{R}^n}\, d^2\sigma(x) \\
&\quad \leqslant \kappa \int_{\Lambda_k S^2} (1+|x|)^{-2-\varepsilon}\, d^2\sigma(x) \\
&\quad = \kappa \Lambda_k^2 \omega_2 (1+\Lambda_k)^{-2-\varepsilon}.
\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \int_{\Lambda_k S^2} \left((\mathbb{K}_{J,z} - \mathbb{H}_z)(x), \frac{x}{\Lambda_k}\right)_{\mathbb{R}^n} d^2\sigma(x) = 0,$$

and

$$\begin{aligned}
2f(z) &= \lim_{k\to\infty} \int_{\Lambda_k S^2} \left(\mathbb{K}_{J,z}(x), \frac{x}{\Lambda_k}\right)_{\mathbb{R}^n} d^2\sigma(x) \\
&= \lim_{k\to\infty} \int_{\Lambda_k S^2} \left(\mathbb{H}_z(x), \frac{x}{\Lambda_k}\right)_{\mathbb{R}^n} d^2\sigma(x) \\
&= \left(\frac{i}{8\pi}\right)(1+z)^{-3/2} \lim_{k\to\infty} \int_{\Lambda_k S^2} \\
&\quad \times \sum_{j=1}^3 \left(\sum_{i_1,i_2=1}^3 \varepsilon_{j i_1 i_2}\, \mathrm{tr}(U(x)(\partial_{i_1}U)(x)(\partial_{i_2}U)(x))\right)\left(\frac{x_j}{\Lambda_k}\right) d^2\sigma(x), \quad (12.17)
\end{aligned}$$

where, for the last integral, we used Proposition 8.13. Theorem 3.4 implies $f(0) = \mathrm{ind}(L)$. In particular, any sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ of positive reals converging to infinity

contains a subsequence $\{\Lambda_{k_\ell}\}_\ell$ such that for that particular subsequence the limit

$$\lim_{\ell\to\infty}\int_{\Lambda_{k_\ell}S^2}\sum_{j,i_1,i_2=1}^{3}\varepsilon_{ji_1i_2}\operatorname{tr}(U(x)(\partial_{i_1}U)(x)(\partial_{i_2}U)(x))\left(\frac{x_j}{\Lambda_{k_\ell}}\right)d^2\sigma(x)$$

exists and equals

$$\frac{2\operatorname{ind}(L)}{[i/(8\pi)]\,(1+z)^{-3/2}}. \tag{12.18}$$

Hence, the limit

$$\lim_{\Lambda\to\infty}\int_{\Lambda S^2}\sum_{j,i_1,i_2=1}^{3}\varepsilon_{ji_1\dots i_2}\operatorname{tr}(U(x)(\partial_{i_1}U)(x)(\partial_{i_2}U)(x))\left(\frac{x_j}{\Lambda}\right)d^2\sigma(x) \tag{12.19}$$

exists and equals the number in (12.18). On the other hand, for $\operatorname{Re}(z) > z_0$ with $z_0 > 0$ sufficiently large (according to Corollary 12.16 $(i)$) the family $\{f_\Lambda\}_{\Lambda>0}$ converges for $\Lambda\to\infty$ on the domain $\mathbb{C}_{\mathrm{Re}>z_0}\cap\Sigma_{\delta,\vartheta}$ if and only if the limit in (12.19) exists. Indeed, this follows from the explicit expression for the limit in (12.17). Therefore, $\{f_\Lambda\}_{\Lambda>0}$ converges in the compact open topology on $\mathbb{C}_{\mathrm{Re}>z_0}\cap\Sigma_{\delta,\vartheta}$. By the local boundedness of the latter family on $\Sigma_{\delta,\vartheta}$, the principle of analytic continuation for analytic functions implies that the latter family actually converges on the domain $\Sigma_{\delta,\vartheta}$ in the compact open topology. In particular,

$$\begin{aligned}&\frac{2f(z)(1+z)^{3/2}}{i/(8\pi)}\\&\quad=\lim_{\Lambda\to\infty}\int_{\Lambda S^2}\sum_{j,i_1,i_2=1}^{3}\varepsilon_{ji_1\dots i_2}\operatorname{tr}(U(x)(\partial_{i_1}U)(x)(\partial_{i_2}U)(x))\left(\frac{x_j}{\Lambda}\right)d^2\sigma(x).\end{aligned}$$

Part $(\delta)$: The Fredholm property of $L$ and $\widetilde{L}$ follows from Lemma 13.3, and the equality $\operatorname{ind}\big(\widetilde{L}\big) = \operatorname{ind}(L)$ follows from Theorem 12.2 and Remark 12.4. The remaining equality in (12.15) follows from part $(\gamma)$ and Theorem 3.4. $\square$

## 13. The proof of Theorem 10.2: The General Case

The strategy to prove Theorem 10.2 for potentials which are only $C^2$ consists in an additional convolution with a suitable mollifier, applying Theorem 10.2 (i.e., Theorem 12.17) for the $C^\infty$-case, and to use suitable perurbation theorems for the Fredholm index. The next result gathers information on mollified functions.

**Proposition 13.1.** *Let $n, d \in \mathbb{N}$, $\Phi \in C_b^1\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$. Assume $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\zeta \geqslant 0$, $\operatorname{supp}(\zeta) \subset B(0,1)$, $\int_{\mathbb{R}^n} \zeta(x)\, d^n x = 1$ and for $\gamma > 0$ define $\zeta_\gamma \coloneqq \gamma^{-n}\zeta\left(1/\gamma\cdot\right)$ and $\Phi_\gamma \coloneqq \zeta_\gamma * \Phi$.*

*$(i)$ For all $0 < \gamma < 1$ and $j \in \{1, \ldots, n\}$ one has*

$$\|\Phi(x) - \Phi_\gamma(x)\| \leqslant \gamma \sup_{y \in B(0,\gamma)} \|\Phi'(x+y)\|, \quad x \in \mathbb{R}^n.$$

*$(ii)$ Assume, in addition, $\Phi \in C_b^2\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ and for some $\varepsilon > 1/2$ and all $\alpha \in \mathbb{N}_0^n$, $|\alpha| = 2$, that for some $\kappa > 0$,*

$$\|\partial^\alpha \Phi(x)\| \leqslant \kappa(1+|x|)^{-1-\varepsilon}, \quad x \in \mathbb{R}^n.$$

*Then for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 2$, $0 < \gamma < 1$,*

$$\|(\partial^\alpha \Phi_\gamma)(x)\| \leqslant \|\partial^{\alpha-\beta}\zeta\|_\infty v_n \gamma^{2-|\alpha|}\kappa(1-\gamma+|x|)^{-1-\varepsilon}, \quad x \in \mathbb{R}^n,$$

*with $v_n$ the $n$-dimensional volume of the unit ball in $\mathbb{R}^n$ and $\beta \in \mathbb{N}_0^n$ such that $(\alpha - \beta) \in \mathbb{N}_0^n$ and $|\beta| = 2$.*

*$(iii)$ If $\Phi(x) = \Phi(x)^*$ for all $x \in \mathbb{R}^n$, then $\Phi_\gamma(x) = \Phi_\gamma(x)^*$ for all $x \in \mathbb{R}^n$, $\gamma > 0$.*

*$(iv)$ If there exist $c > 0$ and $R > 0$ such that $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n\backslash B(0,R)$, then there exists $\gamma_0 > 0$ such that for all $0 < \gamma < \gamma_0$*

$$|\Phi_\gamma(x)| \geqslant (c/2)I_d, \quad x \in \mathbb{R}^n\backslash B(0,R).$$

*Proof.* $(i)$ In order to prove the first inequality, let $x \in \mathbb{R}^n$, $0 < \gamma < 1$. Then one computes

$$\begin{aligned}
\|\Phi(x) - \Phi_\gamma(x)\| &= \left\| \int_{\mathbb{R}^n} \left(\Phi(x) - \Phi(x-y)\right)\zeta_\gamma(y)\, d^n y \right\| \\
&\leqslant \int_{B(0,1)} \|(\Phi(x) - \Phi(x-\gamma y))\|\, \zeta(y)\, d^n y \\
&\leqslant \left( \int_{B(0,1)} \zeta(y)\, d^n y \right) \gamma \sup_{y \in B(0,\gamma)} \|\Phi'(x+y)\|.
\end{aligned}$$

$(ii)$ Let $0 < \gamma < 1$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 2$, and let $\beta \in \mathbb{N}_0^n$ with $|\beta| = 2$ be such that $(\alpha - \beta) \in \mathbb{N}_0^n$. Then for $x \in \mathbb{R}^n$,

$$\begin{aligned}
\|(\partial^\alpha \Phi_\gamma)(x)\| &= \|(\partial^\alpha(\zeta_\gamma * \Phi))(x)\| \\
&= \left\| \left( \left(\partial^{\alpha-\beta}\zeta_\gamma\right) * \left(\partial^\beta \Phi\right) \right)(x) \right\| \\
&= \left\| \int_{\mathbb{R}^n} \left(\partial^{\alpha-\beta}\zeta_\gamma\right)(y) \big(\partial^\beta \Phi\big)(x-y)\, d^n y \right\| \\
&\leqslant \int_{\mathbb{R}^n} \gamma^{-n-|\alpha|+2} \big\| \big(\partial^{\alpha-\beta}\zeta\big)(y/\gamma)(\partial^\beta\Phi)(x-y) \big\|\, d^n y \\
&\leqslant \int_{B(0,1)} \gamma^{2-|\alpha|} \left| \big(\partial^{\alpha-\beta}\zeta\big)(y) \right| \big\| (\partial^\beta \Phi)(x-\gamma y) \big\|\, d^n y
\end{aligned}$$

$$\begin{aligned}&\leqslant \|\partial^{\alpha-\beta}\zeta\|_\infty \int_{B(0,1)} \gamma^{2-|\alpha|}\kappa(1-\gamma+|x|)^{-1-\varepsilon}\, d^n y\\&\leqslant \|\partial^{\alpha-\beta}\zeta\|_\infty v_n \gamma^{2-|\alpha|}\kappa(1-\gamma+|x|)^{-1-\varepsilon}.\end{aligned}$$

$(iii)$ This is clear.

$(iv)$ By part $(i)$, there exists $\gamma_0 > 0$ such that $\sup_{x\in\mathbb{R}^n} \|\Phi(x) - \Phi_\gamma(x)\| < c/2$ for all $0 < \gamma < \gamma_0$. Let $x \in \mathbb{R}^n$ such that $|x| \geqslant R$. From $|\Phi(x)| \geqslant cI_d$ it follows that $\|\Phi(x)^{-1}\| \leqslant 1/c$. Hence, $\left\|\Phi(x)^{-1}\left(\Phi(x) - \Phi_\gamma(x)\right)\right\| \leqslant 1/2$ and therefore,

$$\begin{aligned}\sum_{k=0}^{\infty} &\left(\Phi(x)^{-1}\left(\Phi(x)-\Phi_\gamma(x)\right)\right)^k \Phi(x)^{-1} = \left(1-\Phi(x)^{-1}\left(\Phi(x)-\Phi_\gamma(x)\right)\right)^{-1}\Phi(x)^{-1}\\&= \left(\Phi(x) - \left(\Phi(x)-\Phi_\gamma(x)\right)\right)^{-1} = \Phi_\gamma(x)^{-1}.\end{aligned}$$

Using $\left\|\Phi_\gamma(x)^{-1}\right\| \leqslant c^{-1}\sum_{k=0}^\infty 2^{-k} = 2/c$, one deduces with the help of the spectral theorem that $|\Phi_\gamma(x)| \geqslant (c/2)I_d$. $\square$

*Remark* 13.2. Let $\Phi$ be as in Theorem 10.2. More precisely, let $\Phi \in C_b^2\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ be pointwise self-adjoint, suppose that for some $c > 0$ and $R > 0$, $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n\backslash B(0,R)$, and assume there exists $\varepsilon > 1/2$, such that for all $\alpha \in \mathbb{N}_0^n$ there exists $\kappa > 0$ such that

$$\|\partial^\alpha\Phi(x)\| \leqslant \kappa \begin{cases}(1+|x|)^{-1}, & |\alpha| = 1,\\(1+|x|)^{-1-\varepsilon}, & |\alpha| = 2.\end{cases}$$

By Proposition 13.1, there exists $\gamma_0 > 0$ such that for all $\gamma \in (0,\gamma_0)$, $\Phi_\gamma$ (defined as in Proposition 13.1) satisfies the assumptions imposed on $\Phi$ in Theorem 12.17. Moreover, by Proposition 13.1$\,(i)$, for some $\widetilde{\kappa} > 0$, the estimate

$$\|\partial_j\Phi(x) - \partial_j\Phi_\gamma(x)\| \leqslant \widetilde{\kappa}(1+|x|)^{-1-\varepsilon}$$

holds for all $j \in \{1,\dots,n\}$, $x \in \mathbb{R}^n$, $0 < \gamma < 1$. The latter observation will be used in the proof of the general case of Theorem 10.2. $\diamond$

For the sake of completeness, we shall also state the Fredholm property for $C^2$-potentials:

**Lemma 13.3.** *Let $n, d \in \mathbb{N}$, $L = \mathcal{Q} + \Phi$ as in (7.1) with $\Phi \in C_b^2\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ with $\Phi(x) = \Phi(x)^*$, $x \in \mathbb{R}^n$. Assume that there exist $c > 0$ and $R > 0$ such that $|\Phi(x)| \geqslant cI_d$ for all $x \in \mathbb{R}^n\backslash B(0,R)$, and that $(\partial_j\Phi)(x) \to 0$ as $|x| \to \infty$, $j \in \{1,\dots,n\}$. Then $L$ is a Fredholm operator, and there is $\gamma_0 > 0$ such that for all $\gamma \in (0,\gamma_0)$, $\operatorname{ind}(L) = \operatorname{ind}(L_\gamma)$, where $L_\gamma = \mathcal{Q} + \Phi_\gamma$ with $\Phi_\gamma$ given as in Proposition* 13.1.

*Proof.* By Proposition 13.1, $L$ is a $\mathcal{Q}$-compact perturbation of $L_\gamma$. Moreover, the latter is a Fredholm operator for all $\gamma \in (0,\gamma_0)$ for some $\gamma_0 > 0$ by Proposition 13.1 $\big($guaranteeing that for some $\widetilde{c} > 0$ and $\widetilde{R} > 0$, $|\Phi_\gamma(x)| \geqslant \widetilde{c}I_d$ for all $x \in \mathbb{R}^n\backslash B\big(0,\widetilde{R}\big)\big)$ and Lemma 12.1. Thus, the assertion follows from the invariance of the Fredholm index under relatively compact perurbations. $\square$

We are prepared to conclude the proof of the main theorem also for $C^2$-potentials.

*Proof of Theorem* 10.2, *the nonsmooth case.* Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be as in Proposition 13.1, and define $\Phi_\gamma$ as in the latter proposition. Let $\gamma_0 \in (0,1)$ be as in Proposition

13.1 $(iv)$. As observed in Remark 13.2, $\Phi_\gamma$ satisfies the assumptions imposed on $\Phi$ in Theorem 12.17. Next, by Lemma 13.3, $L_\gamma \coloneqq \mathcal{Q} + \Phi_\gamma$ is Fredholm and $\operatorname{ind}(L) = \operatorname{ind}(L_\gamma)$.

Hence, by the $C^\infty$-version of Theorem 10.2, that is, by Threorem 12.17, one infers that

$$\begin{aligned}\operatorname{ind}(L_\gamma) = \left(\frac{i}{8\pi}\right)^{(n-1)/2} &\frac{1}{[(n-1)/2]!} \lim_{\Lambda\to\infty} \frac{1}{2\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{ji_1\dots i_{n-1}} \\ &\times \int_{\Lambda S^{n-1}} \operatorname{tr}\big(\operatorname{sgn}(\Phi_\gamma(x))(\partial_{i_1}\operatorname{sgn}(\Phi_\gamma))(x) \\ &\qquad \dots(\partial_{i_{n-1}}\operatorname{sgn}(\Phi_\gamma))(x)\big) x_j \, d^{n-1}\sigma(x).\end{aligned}$$

It sufices to shown that the limit $\gamma \to 0$ in the latter expression exists and coincides with the formula asserted. By differentiation, one observes that $\omega\colon \mathbb{R}_{>0} \ni x \mapsto \sqrt{x^2+c} - x$ is decreasing and, denoting $\|\Phi\|_\infty \coloneqq \sup_{x\in\mathbb{R}^n} \|\Phi(x)\|$, one gets $0 < \omega(\|\Phi\|_\infty) \leqslant \omega(\|\Phi(x)\|)$, $x \in \mathbb{R}^n$. Let $0 < \gamma_1 < \omega(\|\Phi\|_\infty)/(4\kappa) \wedge (1/2) \wedge \gamma_0$. For $0 < \gamma < \gamma_1$ and all $x \in \mathbb{R}^n$ one deduces with the help of Proposition 13.1 $(i)$ that

$$\|\Phi(x) - \Phi_\gamma(x)\| \leqslant \gamma_1 2\kappa(1-\gamma_1+|x|)^{-1} \leqslant \omega(\|\Phi\|_\infty)/2.$$

Hence, by Theorem 10.3, one gets for some $K > 0$ and all $x \in \mathbb{R}^n$ with $|x| \geqslant R$,

$$\begin{aligned}\|\operatorname{sgn}(\Phi(x)) - \operatorname{sgn}(\Phi_\gamma(x))\| &\leqslant \sup_{T\in\bigcup_{|x|\geqslant R} B(\Phi(x),\omega((\|\Phi\|_\infty)/2)} \|\operatorname{sgn}'(T)\|\|\Phi(x) - \Phi_\gamma(x)\| \\ &\leqslant \gamma K(1+|x|)^{-1}.\end{aligned}$$

Similarly, Proposition 13.1 $(i)$ implies for some $K' > 0$,

$$\begin{aligned}&\max_{j\in\{1,\dots,n\}} \|\operatorname{sgn}'(\Phi(x))(\partial_j\Phi)(x) - \operatorname{sgn}'(\Phi_\gamma(x))(\partial_j\Phi)(x)\| \\ &\quad\leqslant \max_{j\in\{1,\dots,n\}} \|(\partial_j\Phi)(x)\| \sup_{T\in\bigcup_{|x|\geqslant R} B(\Phi(x),\omega(\|\Phi\|_\infty)/2)} \|\operatorname{sgn}''(T)\|\|\Phi(x) - \Phi_\gamma(x)\| \\ &\quad\leqslant \gamma K'(1+|x|)^{-2}.\end{aligned}$$

For $i_1, \dots, i_{n-1} \in \{1,\dots,n\}$ and $x \in \mathbb{R}^n$, and with the convention $\partial_{i_0} \coloneqq 1$, one gets for some constants $K'', K''' > 0$,

$$\begin{aligned}&\|(\operatorname{sgn}(\Phi(x))(\partial_{i_1}\operatorname{sgn}(\Phi))(x)\dots(\partial_{i_{n-1}}\operatorname{sgn}(\Phi))(x)) \\ &\qquad - (\operatorname{sgn}(\Phi_\gamma(x))(\partial_{i_1}\operatorname{sgn}(\Phi_\gamma))(x)\dots(\partial_{i_{n-1}}\operatorname{sgn}(\Phi_\gamma))(x))\| \\ &\quad\leqslant \sum_{j=0}^{n-1} \bigg\| \prod_{k=0}^{j-1} (\partial_{i_k}\operatorname{sgn}(\Phi))(x)((\partial_{i_j}\operatorname{sgn}(\Phi))(x) - (\partial_{i_j}\operatorname{sgn}(\Phi_\gamma))(x)) \\ &\qquad\qquad \times \prod_{k=j+1}^{n-1} (\partial_{i_k}\operatorname{sgn}(\Phi_\gamma))(x)\bigg\| \\ &\quad\leqslant K''(1+|x|)^{2-n}\bigg(\sum_{j=1}^{n-1} \|(\partial_{i_j}\operatorname{sgn}(\Phi))(x) - (\partial_{i_j}\operatorname{sgn}(\Phi_\gamma))(x)\| \\ &\qquad\qquad\qquad + (1+|x|)^{-1}\|\operatorname{sgn}(\Phi(x)) - \operatorname{sgn}(\Phi_\gamma(x))\|\bigg)\end{aligned}$$

$$
\begin{aligned}
&\leqslant K''(1+|x|)^{2-n}\Bigg(\sum_{j=1}^{n-1}\|\operatorname{sgn}'(\Phi(x))(\partial_{i_j}\Phi)(x)-\operatorname{sgn}'(\Phi_\gamma(x))(\partial_{i_j}\Phi_\gamma)(x)\| \\
&\qquad\qquad\qquad\qquad + \gamma K(1+|x|)^{-2}\Bigg) \\
&\leqslant K''(1+|x|)^{2-n}\Bigg(\sum_{j=1}^{n-1}\|\operatorname{sgn}'(\Phi(x))(\partial_{i_j}\Phi)(x)-\operatorname{sgn}'(\Phi_\gamma(x))(\partial_{i_j}\Phi)(x)\| \\
&\quad + \sum_{j=1}^{n-1}\|\operatorname{sgn}'(\Phi_\gamma(x))(\partial_{i_j}\Phi)(x)-\operatorname{sgn}'(\Phi_\gamma)(x)(\partial_{i_j}\Phi_\gamma)(x)\| + \gamma K(1+|x|)^{-2}\Bigg) \\
&\leqslant K''(1+|x|)^{2-n}\Bigg(\sum_{j=1}^{n-1}\gamma K'(1+|x|)^{-2} \\
&\quad + \sum_{j=1}^{n-1}\sup_{T\in\bigcup_{|x|\geqslant R}B(\Phi(x),\omega(\|\Phi\|_\infty)/2)}\|\operatorname{sgn}'(T)\|\|(\partial_{i_j}\Phi)(x)-(\partial_{i_j}\Phi_\gamma)(x)\| \\
&\quad + \gamma K(1+|x|)^{-2}\Bigg) \\
&\leqslant K'''\gamma(1+|x|)^{1-n-\varepsilon}.
\end{aligned} \tag{13.1}
$$

Next, for $\Lambda > 0$ we define

$$
\phi_\Lambda \coloneqq \frac{1}{\Lambda}\sum_{j,i_1,\dots,i_{n-1}=1}^{n}\varepsilon_{ji_1\dots i_{n-1}} \times \int_{\Lambda S^{n-1}} \operatorname{tr}\big(\operatorname{sgn}(\Phi(x))(\partial_{i_1}\operatorname{sgn}(\Phi(x)))\dots(\partial_{i_{n-1}}\operatorname{sgn}(\Phi(x)))\big)x_j\, d^{n-1}\sigma(x)
$$

and

$$
\phi_\Lambda^\gamma \coloneqq \frac{1}{\Lambda}\sum_{j,i_1,\dots,i_{n-1}=1}^{n}\varepsilon_{ji_1\dots i_{n-1}}\int_{\Lambda S^{n-1}} \operatorname{tr}\big(\operatorname{sgn}(\Phi_\gamma(x)) \times (\partial_{i_1}\operatorname{sgn}(\Phi_\gamma(x)))\dots(\partial_{i_{n-1}}\operatorname{sgn}(\Phi_\gamma(x)))\big)x_j\, d^{n-1}\sigma(x).
$$

It remains to prove that $\{\phi_\Lambda\}_\Lambda$ converges and that its limit coincides with $\operatorname{ind}(L)$. But, with the help of estimate (13.1) one gets

$$
\limsup_{\Lambda\to\infty}|\phi_\Lambda^\gamma-\phi_\Lambda| \leqslant \limsup_{\Lambda\to\infty}\int_{\Lambda S^{n-1}} K'''\gamma(1+|x|)^{1-n-\varepsilon}\frac{|x|}{\Lambda}d^{n-1}\sigma(x) = 0,
$$

which implies the remaining assertion. □

*Remark* 13.4. (*i*) Of course a simple manner in which to invoke less regular potentials is the perturbation with compactly supported potentials. Thus, the above result should be read as $\Phi$ is $C^2$ "in a neighborhood of infinity".

(*ii*) The formula for the Fredholm index suggests that Theorem 10.2 might be weakened in the sense that potentials that are only $C^1$ should lead to the same result. Our method of proof needs that for some $K > 0$ and all $\gamma > 0$ sufficiently small, $\big\|(\partial_{i_j}\Phi)(x)-(\partial_{i_j}\Phi_\gamma)(x)\big\| \leqslant K(1+|x|)^{-1-\varepsilon}$. To prove the latter estimate we need information on the second derivative of $\Phi$. ◇

We conclude with a nontrivial example of the Fredholm index. In view of the discussion in Example 4.8 and the erroneous statement in (1.22) this could be the type of potentials Callias had in mind.

**Example 13.5.** *Let $n = 3$, $\gamma_{1,3}, \gamma_{2,3}, \gamma_{3,3} \in \mathbb{C}^{2\times 2}$ be the corresponding matrices of the Euclidean Dirac Algebra* (*see Appendix* A). *Consider $L = \mathcal{Q}+\Phi$ as in* (7.1) *with $\Phi(x) \coloneqq \sum_{j=1}^{3} \gamma_{j,3} x_j \, |x|^{-1}$, $j \in \{1, 2, 3\}$. Then $\Phi(x)^2 = I_2$ and Theorem* 10.2 *applies. Given formula* (10.1) *for the Fredholm index, a straightforward computation yields*

$$\operatorname{tr}_2 \big(\Phi(x)(\partial_{i_1}\Phi)(x)(\partial_{i_2}\Phi)(x)\big) = \operatorname{tr}_2 \big(\gamma_{j,3}\gamma_{i_1,3}\gamma_{i_2,3} x_j \, |x|^{-3}\big), \quad x \in \mathbb{R}^n \backslash \{0\},$$

*for all $j, i_1, i_2 \in \{1, 2, 3\}$ pairwise distinct, and hence,*

$$\begin{aligned}
\operatorname{ind}(L) &= \frac{i}{16\pi} \lim_{\Lambda\to\infty} \sum_{j,i_1,i_2=1}^{3} \varepsilon_{j i_1 i_2} \frac{1}{\Lambda} \int_{\Lambda S^2} \operatorname{tr}\big(\Phi(x)(\partial_{i_1}\Phi)(x)(\partial_{i_2}\Phi)(x)\big) x_j \, d^2\sigma(x) \\
&= \frac{i}{16\pi} \lim_{\Lambda\to\infty} \sum_{j=1}^{3} \frac{1}{\Lambda} \int_{\Lambda S^2} 4i \frac{x_j}{\Lambda^3} x_j \, d^2\sigma(x) \\
&= \frac{i}{16\pi} \lim_{\Lambda\to\infty} \frac{1}{\Lambda^2} \int_{\Lambda S^2} 4i \, d^2\sigma(x) \\
&= \frac{-1}{4\pi} \int_{S^2} d^2\sigma(x) = -1.
\end{aligned}$$

## 14. A Particular Class of Non-Fredholm Operators $L$ and Their Generalized Witten Index

This section is devoted to a particular non-Fredhom situation and motivated in part by extensions of index theory for a certain class of non-Fredholm operators initiated in [12], [24], [53] (see also [13], [27]). Here we make the first steps in the direction of non-Fredholm operators closely related to the operator $L$ in (6.2) studied by Callias [22] and introduce a generalized Witten index.

We very briefly outline the idea presented in [53]: Let $L$ be a densely defined, closed, linear operator in a Hilbert space $\mathcal{H}$. Assume that

$$\big[(L^*L+z)^{-1}-(LL^*+z)^{-1}\big]\in\mathcal{B}_1(\mathcal{H})$$

for one (and hence for all) $z\in\varrho(-L^*L)\cap\varrho(-LL^*)$, and that the limit

$$\operatorname{ind}_W(L)\coloneqq\lim_{x\downarrow 0_+}x\operatorname{tr}_{\mathcal{H}}\big((L^*L+x)^{-1}-(LL^*+x)^{-1}\big) \tag{14.1}$$

exists. Then $\operatorname{ind}_W(L)$ is called the *Witten index* of $L$. In fact, for the special case of operators in space dimension $n=1$ (with appropriate potential), this limit is easily shown to exist and to assume values in $(1/2)\mathbb{Z}$, see [23]. These examples, however, heavily rely on the fact that the underlying spatial dimension for the operator $L$ equals one.

While the Fredholm index is well-known to be invariant with respect to relatively compact additive perturbations, we emphasize that this cannot hold for the Witten index (cf. [12], [53]). In fact, it can be shown that the Witten index is invariant under additive perturbations that are relatively trace class (among additional conditions, see [53] for details).

We now provide a further generalization of the Witten index adapted to the non-Fredholm operators discussed in this section for odd dimensions $n\geqslant 3$. The abstract set-up reads as follows:

**Definition 14.1.** *Let $L$ be a densely defined, closed linear operator in $\mathcal{H}^m$ for some $m\in\mathbb{N}$. Assume there exist sequences $\{T_\Lambda\}_{\Lambda\in\mathbb{N}}$, $\{S_\Lambda^*\}_{\Lambda\in\mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$ converging to $I_{\mathcal{H}}$ in the strong operator topology, and denote $S_\Lambda\coloneqq S_\Lambda^{**}$, $\Lambda\in\mathbb{N}$. In addition, suppose that the map*

$$\Omega\ni z\mapsto T_\Lambda\operatorname{tr}_m\big((L^*L+z)^{-1}-(LL^*+z)^{-1}\big)S_\Lambda$$

*assumes values in $\mathcal{B}_1(\mathcal{H})$ for some open set $\Omega\subseteq\varrho(-L^*L)\cap\varrho(-LL^*)$ with $(0,\delta)\subseteq\Omega\cap\mathbb{R}$ for some $\delta>0$. Moreover, assume that $\{f_\Lambda\}_{\Lambda\in\mathbb{N}}$, where*

$$f_\Lambda\colon\Omega\ni z\mapsto z\operatorname{tr}_{\mathcal{H}}\big(T_\Lambda\operatorname{tr}_m\big((L^*L+z)^{-1}-(LL^*+z)^{-1}\big)S_\Lambda\big),\quad\Lambda\in\mathbb{N},$$

*converges in the compact open topology as $\Lambda\to\infty$ to some function $f\colon\Omega\to\mathbb{C}$ and that $f(0_+)$ exists. Then we call*

$$\operatorname{ind}_{gW,T,S}(L)\coloneqq f(0_+). \tag{14.2}$$

*the* generalized Witten index *of $L$ (with respect to $T$ and $S$). If $L$ satisfies the assumptions needed for defining* $\operatorname{ind}_{gW,T,S}(L)$*, then we say that $L$* admits a generalized Witten index.

Whenever the sequences $\{T_\Lambda\}_{\Lambda\in\mathbb{N}}$ and $\{S_\Lambda\}_{\Lambda\in\mathbb{N}}$ in Definition 14.1 are clear from the context, we will just write $\operatorname{ind}_{gW}(\cdot)$ instead of $\operatorname{ind}_{gW,T,S}(\cdot)$.

*Remark* 14.2. We briefly elaborate on some properties of the regularized index just defined.

$(i)$ It is easy to see that the generalized Witten index is independent of the chosen $\Omega$. Indeed, the main observation needed is that if $\Omega_1$ and $\Omega_2$ satisfy the requirements imposed on $\Omega$ in Definition 14.1, then so does $\Omega_1 \cap \Omega_2$.

$(ii)$ The generalized Witten index is invariant under unitary equivalence of $\mathcal{H}$. Indeed, let $L$ be a densely defined, closed linear operator in $\mathcal{H}^m$ for which the generalized Witten index exists with respect to $\{T_\Lambda\}_{\Lambda\in\mathbb{N}}$ and $\{S_\Lambda\}_{\Lambda\in\mathbb{N}}$. Let $\mathcal{H}_1$ be another Hilbert space and $U\colon \mathcal{H}_1 \to \mathcal{H}$ an isometric isomorphism. Then

$$\widetilde{L} \coloneqq \begin{pmatrix} U^* & 0 & \cdots & 0 \\ 0 & U^* & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & U^* \end{pmatrix} L \begin{pmatrix} U & 0 & \cdots & 0 \\ 0 & U & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & U \end{pmatrix}$$

admits a generalized Witten index, and

$$\operatorname{ind}_{gW,T,S}(L) = \operatorname{ind}_{gW,U^*TU,U^*SU}\big(\widetilde{L}\big),$$

where, in obvious notation, we used $\operatorname{ind}_{gW,U^*TU,U^*SU}\big(\widetilde{L}\big)$ to denote the generalized Witten index of $\widetilde{L}$ with respect to $\{U^*T_\Lambda U\}_{\Lambda\in\mathbb{N}}$ and $\{U^*S_\Lambda U\}_{\Lambda\in\mathbb{N}}$.

For the proof of $\widetilde{L}$ admitting a generalized Witten index, it suffices to observe that for $\Lambda \in \mathbb{N}$ and $z \in \Omega$,

$$\begin{aligned} &\operatorname{tr}_{\mathcal{H}}\big(T_\Lambda \operatorname{tr}_m\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)S_\Lambda\big) \\ &\quad = \operatorname{tr}_{\mathcal{H}}\big(T_\Lambda UU^* \operatorname{tr}_m\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)UU^*S_\Lambda\big) \\ &\quad = \operatorname{tr}_{\mathcal{H}}\big(T_\Lambda U \operatorname{tr}_m\big((\widetilde{L}^*\widetilde{L}+z)^{-1} - (\widetilde{L}\widetilde{L}^*+z)^{-1}\big)U^*S_\Lambda\big) \\ &\quad = \operatorname{tr}_{\mathcal{H}_1}\big(U^*T_\Lambda U \operatorname{tr}_m\big((\widetilde{L}^*\widetilde{L}+z)^{-1} - (\widetilde{L}\widetilde{L}^*+z)^{-1}\big)U^*S_\Lambda U\big). \end{aligned}$$

$\diamond$

*Remark* 14.3. The definition of the Witten index in (14.1) suggests introducing the spectral shift function $\xi(\,\cdot\,; LL^*, L^*L)$ for the pair of self-adjoint operators $(LL^*, L^*L)$ and hence to express the Witten index as

$$\operatorname{ind}_W(L) = \xi(0_+; LL^*, L^*L), \tag{14.3}$$

employing the fact (see, e.g., [105, Ch. 8]),

$$\operatorname{tr}_{\mathcal{H}}\big(f(L^*L) - f(LL^*)\big) = \int_{[0,\infty)} f'(\lambda)\, \xi(\lambda; LL^*, L^*L)\, d\lambda, \tag{14.4}$$

for a large class of functions $f$. The approach (14.3) in terms of spectral shift functions was introduced in [12], [53] (see also [13], [19], [79, Chs. IX, X], [80]) and independently in [27]. This circle of ideas continues to generate much interest, see, for instance, [23], [24], [52], and the extensive list of references therein. It remains to be seen if this can be applied to the generalized Witten index (14.2). $\diamond$

Next, we will construct non-Fredholm Callias-type operators $L$, which meet the assumptions in the definition for the generalized Witten index, that is, operators $L$ which admit a generalized Witten index. In fact, the theory developed in the previous chapters provides a variety of such examples (cf. the end of this section in Theorem 14.11).

We start with an elementary observation:

**Proposition 14.4.** *Let $n = 2\widehat{n} + 1 \in \mathbb{N}$ odd. Then $Q$ as in (4.1) with $\operatorname{dom}(Q) = H^1(\mathbb{R}^n)^{2^{\widehat{n}}}$ as an operator in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}}$ is non-Fredholm.*

*Proof.* It suffices to observe that the symbol of $Q$ is a continuous function vanishing at 0. □

The fundamental result leading to Theorem 14.11 is contained in the following lemma.

**Lemma 14.5.** *Let $n = 2\widehat{n} + 1 \in \mathbb{N}$ odd, $d \in \mathbb{N}$, $\mathcal{Q}$ as in (6.3), $\Phi \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ pointwise self-adjoint, and let $L = \mathcal{Q} + \Phi$ be as in (7.1). Assume that there exists $P = P^* = P^2 \in \mathbb{C}^{d\times d}\backslash\{0\}$ such that for all $x \in \mathbb{R}^n$,*

$$P\Phi(x) = \Phi(x)P = 0.$$

*Define $\mathcal{P} \coloneqq I_{L^2(\mathbb{R}^n)^{2^{\widehat{n}}}} \otimes P$ and denote $\mathcal{H}_{\mathcal{P}} \coloneqq L^2(\mathbb{R}^n)^{2^{\widehat{n}}} \otimes \operatorname{ran}(P)$. Then $L$ and $L^*$ leave the space $\mathcal{H}_P$ invariant. Moreover, $L$ is unitarily equivalent to*

$$\begin{pmatrix} \iota_P^* \mathcal{Q} \iota_P & 0 \\ 0 & \iota_{P_\perp}^* \mathcal{Q} \iota_{P_\perp} + \iota_{P_\perp}^* \Phi \iota_{P_\perp} \end{pmatrix}$$

*with $\iota_P$ and $\iota_{P_\perp}$ the canonical embeddings from $\mathcal{H}_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}}^\perp$ into $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$, respectively.*

*Remark* 14.6. Recalling $Q$ given as in (4.1), we claim that in the situation of Lemma 14.5,

$$\iota_P^* \mathcal{Q} \iota_P = Q \otimes I_{\operatorname{ran} P}.$$

Indeed, equality is plain when applied to $C_0^\infty$-functions and thus the general case follows by a closure argument. For the closedness of $\iota_P^* \mathcal{Q} \iota_P$ we use Lemma 14.5: $L$ is closed and, by unitary equivalence, so is

$$\begin{pmatrix} \iota_P^* \mathcal{Q} \iota_P & 0 \\ 0 & \iota_{P_\perp}^* \mathcal{Q} \iota_{P_\perp} + \iota_{P_\perp}^* \Phi \iota_{P_\perp} \end{pmatrix}. \tag{14.5}$$

Hence, the diagonal entries of the closed block operator matrix in (14.5), and thus $\iota_P^* \mathcal{Q} \iota_P$, are closed. ◇

In order to prove Lemma 14.5, we invoke some auxiliary results of a general nature. The first two (Lemmas 14.7 and 14.8) are concerned with commutativity properties of the operator $\mathcal{Q}$.

**Lemma 14.7.** *Let $n, m \in \mathbb{N}$, $P \in \mathbb{C}^{m\times m}$, $j \in \{1, \dots, n\}$. Then*

$$(I_{L^2(\mathbb{R}^n)} \otimes P)\partial_j \subseteq \partial_j (I_{L^2(\mathbb{R}^n)} \otimes P),$$

*where $\partial_j \colon H_j^1(\mathbb{R}^n)^m \subseteq L^2(\mathbb{R}^n)^m \to L^2(\mathbb{R}^n)^m$ is the distributional derivative with respect to the $j$th variable and, $H_j^1(\mathbb{R}^n)$ is the space of $L^2$-functions whose derivative with respect to the $j$th variable can be represented as an $L^2$-function.*

*Proof.* Clearly,

$$(I_{L^2(\mathbb{R}^n)} \otimes P)\partial_j \phi = \partial_j (I_{L^2(\mathbb{R}^n)} \otimes P)\phi, \quad \phi \in C_0^\infty(\mathbb{R}^n)^m.$$

Next, the operator $\partial_j (I_{L^2(\mathbb{R}^n)} \otimes P)$ is closed, hence,

$$(I_{L^2(\mathbb{R}^n)} \otimes P)\partial_j \subseteq \overline{(I_{L^2(\mathbb{R}^n)} \otimes P)\partial_j} \subseteq \partial_j (I_{L^2(\mathbb{R}^n)} \otimes P),$$

yields the assertion. □

**Lemma 14.8.** *Let $n, d \in \mathbb{N}$, $n = 2\widehat{n}$ or $n = 2\widehat{n}+1$ for some $\widehat{n} \in \mathbb{N}$. Let $\mathcal{Q}$ as in* (6.3) *(defined in $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$). Let $P \in \mathbb{C}^{d\times d}$ and denote $\mathcal{P} := I_{L^2(\mathbb{R}^n)^{2^{\widehat{n}}}} \otimes P$. Then,*

$$\mathcal{P}\mathcal{Q} \subseteq \mathcal{Q}\mathcal{P}.$$

*Proof.* We note that for all $j \in \{1, \ldots, n\}$ and $\gamma_{j,n}$ as in Section A, $\gamma_{j,n}\mathcal{P} = \mathcal{P}\gamma_{j,n}$, where we viewed $\gamma_{j,n} \in \mathcal{B}(L^2(\mathbb{R}^n)^{2^{\widehat{n}}})$. Hence, using $\operatorname{dom}(\mathcal{Q}) = \bigcap_{j=1}^n \operatorname{dom}(\partial_j)$, Lemma 14.7 implies

$$\begin{aligned}\mathcal{P}\mathcal{Q} &= \mathcal{P}\sum_{j=1}^n \gamma_{j,n}\partial_j = \sum_{j=1}^n \mathcal{P}\gamma_{j,n}\partial_j \\ &= \sum_{j=1}^n \gamma_{j,n}\mathcal{P}\partial_j \subseteq \sum_{j=1}^n \gamma_{j,n}\partial_j\mathcal{P} = \mathcal{Q}\mathcal{P}. \qquad \square\end{aligned}$$

Before turning to the proof of Lemma 14.5, we recall a general result on the representability of operators as block operator matrices (the same calculus has also been employed in [84, Lemma 3.2]):

**Lemma 14.9.** *Let $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection, $W\colon D(W) \subseteq \mathcal{H} \to \mathcal{H}$ closed and linear. Assume that*

$$PW \subseteq WP \text{ and } (I_{\mathcal{H}} - P)W \subseteq W(I_{\mathcal{H}} - P).$$

*Denote by $\iota_P\colon \operatorname{ran}(P) \to \mathcal{H}$ and $\iota_{P_\perp}\colon \ker(P) \to \mathcal{H}$ the canonical embeddings, respectively. Then $W$ is unitarily equivalent to a block operator matrix. More precisely,*

$$\begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} W \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} = \begin{pmatrix} \iota_P^* W \iota_P & 0 \\ 0 & \iota_{P_\perp}^* W \iota_{P_\perp} \end{pmatrix} \tag{14.6}$$

*with $\iota_P^* W \iota_P$ and $\iota_{P_\perp}^* W \iota_{P_\perp}$ closed linear operators.*

*Proof.* First, one observes that the operators $\begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix}$ and $\begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix}$ are unitary and inverses of each other. Moreover, it is plain that a block diagonal operator matrix is closed if and only if its diagonal entries are closed. Thus, as $W$ is closed by hypothesis, it suffices to prove equality (14.6). One notes that $P = \iota_P\iota_P^*$ and similarly, $(I_{\mathcal{H}} - P) = \iota_{P_\perp}\iota_{P_\perp}^*$, and hence computes,

$$\begin{aligned}
W &= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} W \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} \\
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \\ \iota_{P_\perp}^* W \end{pmatrix} \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} \\
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \iota_P \iota_P^* W \\ \iota_{P_\perp}^* \iota_{P_\perp} \iota_{P_\perp}^* W \end{pmatrix} \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} \\
&\subseteq \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \iota_P \iota_P^* \\ \iota_{P_\perp}^* W \iota_{P_\perp} \iota_{P_\perp}^* \end{pmatrix} \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} \\
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \iota_P \iota_P^* \iota_P & 0 \\ 0 & \iota_{P_\perp}^* W \iota_{P_\perp} \iota_{P_\perp}^* \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix} \\
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \iota_P & 0 \\ 0 & \iota_{P_\perp}^* W \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* \\ \iota_{P_\perp}^* \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \iota_P \iota_P^* \\ \iota_{P_\perp}^* W \iota_{P_\perp} \iota_{P_\perp}^* \end{pmatrix} \\
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \iota_P \iota_P^* \big(\iota_P \iota_P^* + \iota_{P_\perp} \iota_{P_\perp}^*\big) \\ \iota_{P_\perp}^* W \iota_{P_\perp} \iota_{P_\perp}^* \big(\iota_P \iota_P^* + \iota_{P_\perp} \iota_{P_\perp}^*\big) \end{pmatrix} \\
&= \begin{pmatrix} \iota_P & \iota_{P_\perp} \end{pmatrix} \begin{pmatrix} \iota_P^* W \big(\iota_P \iota_P^* + \iota_{P_\perp} \iota_{P_\perp}^*\big) \\ \iota_{P_\perp}^* W \big(\iota_P \iota_P^* + \iota_{P_\perp} \iota_{P_\perp}^*\big) \end{pmatrix} \\
&\subseteq W,
\end{aligned}
$$

concluding the proof. □

At this instant we are in a position to prove Lemma 14.5.

*Proof of Lemma* 14.5. By Lemma 14.8,

$$
\mathcal{P}L \subseteq L\mathcal{P} \ \text{ and } \ (I_{L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}} - \mathcal{P})L \subseteq L(I_{L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}} - \mathcal{P}).
$$

Hence, by Lemma 14.9, $L$ is unitarily equivalent to

$$
\begin{pmatrix} \iota_P^* L \iota_P & 0 \\ 0 & \iota_{P_\perp}^* L \iota_{P_\perp}. \end{pmatrix}
$$

The assertion, thus, follows from $\iota_P^* \Phi \iota_P = 0$ (valid by hypothesis). □

From Proposition 14.4 and Lemma 14.5 one infers the following result.

**Corollary 14.10.** *Let $n = 2\widehat{n}+1 \in \mathbb{N}$ odd, $d \in \mathbb{N}$, assume $\mathcal{Q}$ is given by* (6.3), *$\Phi \in C_b^\infty\big(\mathbb{R}^n; \mathbb{C}^{d\times d}\big)$ pointwise self-adjoint, and let $L = \mathcal{Q}+\Phi$ be as in* (7.1). *Assume that there exists $P = P^* = P^2 \in \mathbb{C}^{d\times d}\backslash\{0\}$ such that*

$$
P\Phi(x) = \Phi(x)P = 0, \quad x \in \mathbb{R}^n.
$$

*Then $L$ is non-Fredholm.*

*Proof.* Define $\mathcal{H}_P := L^2(\mathbb{R}^n; \mathbb{C}^{2^{\widehat{n}}} \otimes \operatorname{ran}(P))$, denote the embedding from $\mathcal{H}_P$ into $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ by $\iota_P$, and denote the embedding from $\mathcal{H}_P^\perp$ into $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$ by $\iota_{P_\perp}$. By Lemma 14.5, the operator $L$ is unitarily equivalent to

$$
\begin{pmatrix} \iota_P^* \mathcal{Q} \iota_P & 0 \\ 0 & \iota_{P_\perp}^* \mathcal{Q} \iota_{P_\perp} + \iota_{P_\perp}^* \Phi \iota_{P_\perp} \end{pmatrix},
$$

which by Remark 14.6 may also be written as

$$
\begin{pmatrix} Q \otimes I_{\operatorname{ran} P} & 0 \\ 0 & Q \otimes I_{\ker P} + \iota_{P_\perp}^* \Phi \iota_{P_\perp} \end{pmatrix}.
$$

In particular,

$$
\sigma(L) = \sigma\big(Q \otimes I_{\operatorname{ran} P}\big) \cup \sigma\big(Q \otimes I_{\ker P} + \iota_{P_\perp}^* \Phi \iota_{P_\perp}\big).
$$

Since $\operatorname{ran}(P)$ is at least one-dimensional, it follows from Proposition 14.4 that $0 \in \sigma_{\mathrm{ess}}\big(Q \otimes I_{\operatorname{ran} P}\big)$. Hence, $0 \in \sigma_{\mathrm{ess}}(L)$, implying that $L$ is non-Fredholm. □

We conclude this section with non-trivial examples illustrating the generalized Witten index introduced in this section:

**Theorem 14.11.** *Assume Hypothesis* 12.6 *and let* $U \in C_b^\infty\big(\mathbb{R}^n;\mathbb{C}^{d\times d}\big)$ *be a* $\tau$-*admissible potential. Let* $\ell \in \mathbb{N}$ *and define*

$$\Phi\colon \mathbb{R}^n \to \mathbb{C}^{(d+\ell)\times(d+\ell)}, \quad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & U(x) \end{pmatrix}.$$

*Let* $L \coloneqq \mathcal{Q} + \Phi$, *as in* (6.2). *Then the following assertions* ($\alpha$)–($\delta$) *hold:*

($\alpha$) *For all* $\Lambda > 0$, *the family*

$$\Sigma_{0,\vartheta} \ni z \mapsto z\chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}(d+\ell)}\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big) \in \mathcal{B}_1(L^2(\mathbb{R}^n)) \tag{14.7}$$

*is analytic.*

($\beta$) *The family* $\{f_\Lambda\}_{\Lambda>0}$ *of analytic functions*

$$f_\Lambda\colon \Sigma_{0,\vartheta} \ni z \mapsto \operatorname{tr}\big(z\chi_\Lambda \operatorname{tr}_{2^{\widehat{n}}(d+\ell)}((L^*L+z)^{-1} - (LL^*+z)^{-1})\big) \tag{14.8}$$

*is locally bounded* (*see* (8.1)).

($\gamma$) *The limit* $f \coloneqq \lim_{\Lambda\to\infty} f_\Lambda$ *exists in the compact open topology and satisfies for all* $z \in \Sigma_{0,\vartheta}$,

$$\begin{aligned} f(z) = c_n (1+z)^{-n/2} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{j i_1 \dots i_{n-1}} \\ \times \int_{\Lambda S^{n-1}} \operatorname{tr}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x)) x_j \, d^{n-1}\sigma(x), \end{aligned} \tag{14.9}$$

*where*

$$c_n \coloneqq \frac{1}{2}\left(\frac{i}{8\pi}\right)^{(n-1)/2} \frac{1}{[(n-1)/2]!}.$$

($\delta$) $L$ *is non-Fredholm, it admits a generalized Witten index, given by*

$$\begin{aligned} \operatorname{ind}_{gW}(L) &= f(0_+) \\ &= c_n \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \sum_{j,i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{j i_1 \dots i_{n-1}} \\ &\quad \times \int_{\Lambda S^{n-1}} \operatorname{tr}(U(x)(\partial_{i_1}U)(x)\dots(\partial_{i_{n-1}}U)(x)) x_j \, d^{n-1}\sigma(x), \end{aligned} \tag{14.10}$$

*which is actually an integer as it coincides with the Fredholm index of* $\widetilde{L} \coloneqq \mathcal{Q} + U$ *in* $L^2(\mathbb{R}^n)^{2^{\widehat{n}}d}$, *that is,*

$$\operatorname{ind}_{gW}(L) = \operatorname{ind}\big(\widetilde{L}\big). \tag{14.11}$$

*Proof.* The proof rests on Theorem 12.17, Lemma 14.5, Remark 14.6, and specifically, for the assertion that $L$ is non-Fredholm, on Corollary 14.10. Indeed, invoking Lemma 14.5 and Remark 14.6 with

$$P = \begin{pmatrix} I_\ell & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{(d+\ell)\times(d+\ell)},$$

one computes, recalling $\widetilde{L} = \mathcal{Q} + U$,

$$\begin{aligned} L^*L &= (-\mathcal{Q} + \Phi)(\mathcal{Q} + \Phi) \\ &= \begin{pmatrix} -\mathcal{Q} & 0 \\ 0 & -\mathcal{Q} + U \end{pmatrix} \begin{pmatrix} \mathcal{Q} & 0 \\ 0 & \mathcal{Q} + U \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -\Delta & 0 \\ 0 & \widetilde{L}^*\widetilde{L} \end{pmatrix}.$$

A similar computation applies to $LL^*$. One deduces for $z \in \mathbb{C}_{\mathrm{Re}>0}$,

$$\begin{aligned}
&((L^*L+z)^{-1} - (LL^*+z)^{-1}) \\
&\quad = \begin{pmatrix} (-\Delta+z)^{-1} & 0 \\ 0 & (\widetilde{L}^*\widetilde{L}+z)^{-1} \end{pmatrix} - \begin{pmatrix} (-\Delta+z)^{-1} & 0 \\ 0 & (\widetilde{L}\widetilde{L}^*+z)^{-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 & 0 \\ 0 & (\widetilde{L}^*\widetilde{L}+z)^{-1} - (\widetilde{L}\widetilde{L}^*+z)^{-1} \end{pmatrix}.
\end{aligned}$$

Thus,

$$\operatorname{tr}_{2^{\widehat{n}}(d+\ell)}\big((L^*L+z)^{-1} - (LL^*+z)^{-1})\big) = \operatorname{tr}_{2^{\widehat{n}}d}\big((\widetilde{L}^*\widetilde{L}+z)^{-1} - (\widetilde{L}\widetilde{L}^*+z)^{-1}\big).$$

Hence, the assertions (14.7)–(14.10) indeed follow from Theorem 12.17 applied to $\widetilde{L}$. □

## Appendix A. Construction of the Euclidean Dirac Algebra

For a concise presentation of the construction of the Euclidean Dirac algebra as a specific case of Clifford algebras, see, for instance, [93, Chapter 11].

**Definition A.1.** *Given two matrices* $A = (a_{ij})_{i,j\in\{1,\ldots,n\}} \in \mathbb{C}^{n\times n}$ *and* $B = (b_{ij})_{i,j\in\{1,\ldots,m\}} \in \mathbb{C}^{m\times m}$*, one defines their* Kronecker product $A \circ B$ *by*

$$\begin{aligned} A\circ B &:= \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & \ddots & & \vdots \\ \vdots & & & \\ a_{n1}B & \cdots & & a_{nn}B \end{pmatrix} \\ &= \left(a_{\lceil \frac{p}{m}\rceil \lceil \frac{q}{m}\rceil} b_{(((p-1) \mod m)+1)(((q-1) \mod m)+1)}\right)_{p,q\in\{1,\ldots,mn\}} \in \mathbb{C}^{nm\times nm}, \end{aligned}$$

*where* $\lceil x\rceil := \min\{z\in\mathbb{Z}\,|\,z\geqslant x\}$ *for all* $x\in\mathbb{R}$ *and* $k \bmod \ell$ *denotes the nonnegative integer* $j\in\{0,\ldots,\ell-1\}$ *such that* $k-j$ *is divisible by* $\ell$*, with* $\ell,k\in\mathbb{Z}$*.*

**Proposition A.2.** *Let* $n,m,\ell,k\in\mathbb{N}$*,* $A\in\mathbb{C}^{n\times n}$*,* $B\in\mathbb{C}^{m\times m}$*,* $C\in\mathbb{C}^{\ell\times\ell}$*,* $D\in\mathbb{C}^{k\times k}$*. Then one concludes that*

$$\begin{aligned} &A\circ(B\circ C) = (A\circ B)\circ C, \\ &(A\circ B)^* = A^*\circ B^*, \\ &\operatorname{tr}(A\circ B) = \operatorname{tr}(A)\operatorname{tr}(B), \\ &\text{if } n=m \text{ and } \ell=k \text{ then, } AB\circ CD = (A\circ C)(B\circ D). \end{aligned}$$

*Proof.* We only sketch a proof for the first assertion. It boils down to the following equations,

$$\begin{aligned} \left\lceil \frac{\lceil \frac{j}{k}\rceil}{m}\right\rceil &= \left\lceil \frac{j}{m\,k}\right\rceil, \\ \left(\left(\left\lceil\frac{j}{k}\right\rceil - 1\right) \mod m\right) + 1 &= \left\lceil \frac{(j-1 \mod mk)+1}{k}\right\rceil, \\ (j-1 \mod mk) \mod k = j-1 &\mod k, \quad j\in\{1,\ldots,mnk\}. \end{aligned}$$

The expressions on the left-hand side correspond to the indices of the entries of $A, B$ and $C$, respectively, in $(A\circ B)\circ C$ and, similarly, the expressions on the right-hand sides correspond to the respective indices of the entries of $A$, $B$ and $C$ in $A\circ(B\circ C)$. □

**Definition A.3.** *Introduce the Pauli matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix},$$

*in addition, define*

$$\gamma_{1,2} := \sigma_1, \quad \gamma_{2,2} := \sigma_2.$$

*Let* $\widehat{n}\in\mathbb{N}$*. Recursively, one sets*

$$\begin{aligned} \gamma_{k,2\widehat{n}+1} &:= \gamma_{k,2\widehat{n}}, \quad k\in\{1,\ldots,2\widehat{n}\}, \\ \gamma_{2\widehat{n}+1,2\widehat{n}+1} &:= (-i)^{\widehat{n}}\,\gamma_{1,2\widehat{n}}\cdots\gamma_{2\widehat{n},2\widehat{n}}, \end{aligned}$$

*and*

$$\begin{aligned}\gamma_{k,2\widehat{n}+2} &:= \sigma_1 \circ \gamma_{k,2\widehat{n}}, \quad k \in \{1,\dots,2\widehat{n}\},\\ \gamma_{2\widehat{n}+1,2\widehat{n}+2} &:= i^{\widehat{n}}\sigma_1 \circ (\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}}),\\ \gamma_{2\widehat{n}+2,2\widehat{n}+2} &:= \sigma_2 \circ I_{2^{\widehat{n}}},\end{aligned}$$

*with* $I_r$ *the identity matrix in* $\mathbb{C}^r$, $r \in \mathbb{N}$.

*Remark* A.4. By induction, one obtains

$$\gamma_{k,2\widehat{n}}, \gamma_{k,2\widehat{n}+1}, \gamma_{2\widehat{n}+1,2\widehat{n}+1} \in \mathbb{C}^{2^{\widehat{n}}\times 2^{\widehat{n}}}, \quad k \in \{1,\dots,2\widehat{n}\}. \tag{A.1}$$

$\diamond$

**Lemma A.5.** *Let* $\gamma_1,\dots,\gamma_k \in \mathcal{B}(\mathcal{K})$ *for some Hilbert space* $\mathcal{K}$ *and such that for all* $j,k \in \{1,\dots,k\}, j \neq k$, *one has* $\gamma_j\gamma_k + \gamma_k\gamma_j = 0$. *Then*

$$\gamma_k\gamma_{k-1}\cdots\gamma_1 = (-1)^{k(k-1)/2}\gamma_1\gamma_2\cdots\gamma_k.$$

*Proof.* The assertion being obvious for $k = 1$, we assume that the assertion of the lemma holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\gamma_{k+1}\gamma_k\gamma_{k-1}\cdots\gamma_1 &= (-1)^k\gamma_k\gamma_{k-1}\cdots\gamma_1\gamma_{k+1}\\ &= (-1)^{[k(k-1)/2]+k}\gamma_1\gamma_2\cdots\gamma_k\gamma_{k+1}\\ &= (-1)^{k(k+1)/2}\gamma_1\gamma_2\cdots\gamma_k\gamma_{k+1}.\end{aligned}$$

$\square$

**Corollary A.6.** *For all* $k,l \in \{1,\dots,n\}$, $n \in \mathbb{N}_{\geqslant 2}$, *one has*

$$\gamma_{k,n}\gamma_{l,n} + \gamma_{l,n}\gamma_{k,n} = 2\delta_{kl}I_{2^{\widehat{n}}},$$

*where* $\gamma_{j,n}$ *is given in Definition* A.3, $j \in \{1,\dots,n\}$, *and* $\widehat{n} \in \mathbb{N}$ *is such that* $n = 2\widehat{n}$ *or* $n = 2\widehat{n}+1$.

*Proof.* The assertion holds for $n = 2$. Assume that the assertion is valid for $n = 2\widehat{n}$ for some $\widehat{n} \in \mathbb{N}$. Then Lemma A.5 implies

$$\begin{aligned}\gamma_{2\widehat{n}+1,2\widehat{n}+1}^2 &= (-i)^{2\widehat{n}}(\gamma_{1,2\widehat{n}}\cdots\gamma_{2\widehat{n},2\widehat{n}})(\gamma_{1,2\widehat{n}}\cdots\gamma_{2\widehat{n},2\widehat{n}})\\ &= (-1)^{\widehat{n}}(\gamma_{1,2\widehat{n}}\cdots\gamma_{2\widehat{n},2\widehat{n}})(-1)^{2\widehat{n}(2\widehat{n}-1)/2}(\gamma_{2\widehat{n},2\widehat{n}}\cdots\gamma_{1,2\widehat{n}})\\ &= (-1)^{\widehat{n}+2\widehat{n}^2-\widehat{n}}I_{2^{\widehat{n}}} = I_{2^{\widehat{n}}}.\end{aligned}$$

For $k \in \{1,\dots,2\widehat{n}-1\}$ one computes

$$\begin{aligned}\gamma_{k,2\widehat{n}+1}\gamma_{2\widehat{n}+1,2\widehat{n}+1} &= \gamma_{k,2\widehat{n}}(-i)^{\widehat{n}}(\gamma_{1,2\widehat{n}}\cdots\gamma_{2\widehat{n},2\widehat{n}})\\ &= (-1)^{2\widehat{n}-1}(-i)^{\widehat{n}}(\gamma_{1,2\widehat{n}}\cdots\gamma_{2\widehat{n},2\widehat{n}})\gamma_{k,2\widehat{n}}\\ &= -\gamma_{2\widehat{n}+1,2\widehat{n}+1}\gamma_{k,2\widehat{n}+1}.\end{aligned}$$

Hence, the assertion is established for $\gamma_{k,2\widehat{n}+1}$, $k \in \{1,\dots,2\widehat{n}+1\}$.

For $k,l \in \{1,\dots,2\widehat{n}\}$ one computes with the help of Proposition A.2,

$$\begin{aligned}\gamma_{k,2\widehat{n}+2}\gamma_{l,2\widehat{n}+2} + \gamma_{l,2\widehat{n}+2}\gamma_{k,2\widehat{n}+2} &= (\sigma_1 \circ \gamma_{k,2\widehat{n}})(\sigma_1 \circ \gamma_{l,2\widehat{n}})\\ &\quad + (\sigma_1 \circ \gamma_{l,2\widehat{n}})(\sigma_1 \circ \gamma_{k,2\widehat{n}})\\ &= \sigma_1^2 \circ \gamma_{k,2\widehat{n}}\gamma_{l,2\widehat{n}} + \sigma_1^2 \circ \gamma_{l,2\widehat{n}}\gamma_{k,2\widehat{n}}\\ &= I_2 \circ (\gamma_{k,2\widehat{n}}\gamma_{l,2\widehat{n}} + \gamma_{l,2\widehat{n}}\gamma_{k,2\widehat{n}})\\ &= I_2 \circ 2\delta_{kl}I_{2^{\widehat{n}}} = 2\delta_{kl}I_{2^{\widehat{n}+1}}.\end{aligned}$$

One observes that

$$\begin{aligned}\gamma_{2\widehat{n}+1,2\widehat{n}+2}^2 &= \left(i^{\widehat{n}}\right)^2 \sigma_1^2 \circ (\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}})^2 \\ &= \left(i^{\widehat{n}}\right)^2 \sigma_1^2 \circ (-i)^{-2\widehat{n}}\, I_{2^{\widehat{n}}} = \left(i^{\widehat{n}}\right)^2 \sigma_1^2 \circ (-1)^{-2\widehat{n}} \left(i^{\widehat{n}}\right)^{-2} I_{2^{\widehat{n}}} = I_{2^{\widehat{n}+1}},\end{aligned}$$

using $\gamma_{2\widehat{n}+1,2\widehat{n}+1}^2 = I_{2^{\widehat{n}}}$. Moreover, $\gamma_{2\widehat{n}+2,2\widehat{n}+2}^2 = \sigma_2^2 \circ I_{2^{\widehat{n}}} = I_{2^{\widehat{n}+1}}$. In addition, one notes that

$$\begin{aligned}\gamma_{2\widehat{n}+2,2\widehat{n}+2}\gamma_{2\widehat{n}+1,2\widehat{n}+2} &= \sigma_2 i^{\widehat{n}}\sigma_1 \circ \gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}} \\ &= -i^{\widehat{n}}\sigma_1\sigma_2 \circ \gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}} = -\gamma_{2\widehat{n}+1,2\widehat{n}+2}\gamma_{2\widehat{n}+2,2\widehat{n}+2}, \\ \gamma_{2\widehat{n}+2,2\widehat{n}+2}\gamma_{k,2\widehat{n}+2} &= \sigma_2\sigma_1 \circ \gamma_{k,\widehat{n}} = -\gamma_{k,2\widehat{n}+2}\gamma_{2\widehat{n}+2,2\widehat{n}+2},\end{aligned}$$

and

$$\begin{aligned}\gamma_{2\widehat{n}+1,2\widehat{n}+2}\gamma_{k,2\widehat{n}+2} &= i^{\widehat{n}}\sigma_1\sigma_1 \circ \gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}}\gamma_{k,2\widehat{n}} \\ &= \sigma_1 i^{\widehat{n}}\sigma_1 \circ (-1)^{2\widehat{n}-1}\gamma_{k,2\widehat{n}}\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}} \\ &= -\gamma_{k,2\widehat{n}+2}\gamma_{2\widehat{n}+1,2\widehat{n}+2}\end{aligned}$$

for all $k \in \{1, \dots, 2\widehat{n}\}$, implying the assertion. □

**Corollary A.7.** *For all $k \in \mathbb{N}$, $n \in \mathbb{N}_{\geqslant 2}$, and $k \leqslant n$, one has*

$$\gamma_{k,n}^* = \gamma_{k,n},$$

*where $\gamma_{j,n}$ is given in Definition* A.3*, $j \in \{1, \dots, n\}$.*

*Proof.* We will proceed by induction. Before doing so, we note that due to Corollary A.6 and Lemma A.5, for all $k \in \{1, \dots, n\}$,

$$\gamma_{k,n}\gamma_{k-1,n} \cdots \gamma_{1,n} = (-1)^{k(k-1)/2}\gamma_{1,n}\gamma_{2,n} \cdots \gamma_{k,n}.$$

One observes that $\gamma_{1,2}$ and $\gamma_{2,2}$ are self-adjoint. We assume that $\gamma_{k,2\widehat{n}}$ is self-adjoint for all $k \in \{1, \dots, 2\widehat{n}\}$ for some $\widehat{n} \in \mathbb{N}$. The only matrices not obviously self-adjoint using the induction hypothesis and Proposition A.2 are $\gamma_{2\widehat{n}+1,2\widehat{n}+2}$ and $\gamma_{2\widehat{n}+1,2\widehat{n}+1}$. Since the proof for either case follows along similar lines, it suffices to prove the self-adjointness of $\gamma_{2\widehat{n}+1,2\widehat{n}+2}$. For this purpose one computes,

$$\begin{aligned}\gamma_{2\widehat{n}+1,2\widehat{n}+2}^* &= \left(i^{\widehat{n}}\sigma_1 \circ (\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}})\right)^* \\ &= i^{\widehat{n}}(-1)^{\widehat{n}}\sigma_1^* \circ (\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}})^* \\ &= i^{\widehat{n}}(-1)^{\widehat{n}}\sigma_1 \circ (\gamma_{2\widehat{n},2\widehat{n}} \cdots \gamma_{1,2\widehat{n}}) \\ &= i^{\widehat{n}}(-1)^{\widehat{n}+[2\widehat{n}(2\widehat{n}-1)/2]}\sigma_1 \circ (\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}}) \\ &= i^{\widehat{n}}(-1)^{\widehat{n}+2\widehat{n}^2-\widehat{n}}\sigma_1 \circ (\gamma_{1,2\widehat{n}} \cdots \gamma_{2\widehat{n},2\widehat{n}}) \\ &= \gamma_{2\widehat{n}+1,2\widehat{n}+2}.\end{aligned}$$

□

Next, we proceed to establish the following result on traces:

**Proposition A.8.** *Let $\widehat{n} \in \mathbb{N}$ and suppose that $\gamma_{j,2\widehat{n}}$, $\gamma_{j',2\widehat{n}+1}$, $j \in \{1, \dots, 2\widehat{n}\}$, $j' \in \{1, \dots, 2\widehat{n}+1\}$, are given as in Definition* A.3*. Then,*

$$\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1} \cdots \gamma_{i_{2k+1},2\widehat{n}+1}\right) = 0, \text{ if } i_1, \dots, i_{2k+1} \in \{1, \dots, 2\widehat{n}+1\} \text{ and } k < \widehat{n},$$

$$\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}} \cdots \gamma_{i_{2k+1},2\widehat{n}}\right) = 0, \text{ if } i_1, \dots, i_{2k+1} \in \{1, \dots, 2\widehat{n}\} \text{ and } k \in \mathbb{N},$$

$$\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1} \cdots \gamma_{i_{2\widehat{n}+1},2\widehat{n}+1}\right) = (2i)^{\widehat{n}}\varepsilon_{i_1 \cdots i_{2\widehat{n}+1}}, \text{ if } i_1, \dots, i_{2\widehat{n}+1} \in \{1, \dots, 2\widehat{n}+1\},$$

*where $\varepsilon_{i_1\cdots i_{2\widehat{n}+1}}$ is the fully anti-symmetric symbol in $2\widehat{n}+1$ dimensions, that is, $\varepsilon_{i_1\ldots i_{2\widehat{n}+1}}=0$ whenever $|\{i_1,\ldots,i_{2\widehat{n}+1}\}|<2\widehat{n}+1$ and if $\pi\colon\{1,\ldots,2\widehat{n}+1\}\to\{1,\ldots,2\widehat{n}+1\}$ is bijective, then $\varepsilon_{\pi(1)\cdots\pi(2\widehat{n}+1)}=\operatorname{sgn}(\pi)$.*

*Proof.* The first formula can be seen as follows. Since $k<\widehat{n}$, there exists $i\in\{1,\ldots,2\widehat{n}+1\}\backslash\{i_1,\ldots,i_{2k+1}\}$, and one computes

$$\begin{aligned}
&\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\right)=\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}^2\right)\\
&\quad=\operatorname{tr}\left(\gamma_{i,2\widehat{n}+1}\gamma_{i_1,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\right)\\
&\quad=-\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\right)\\
&\quad=\ldots=(-1)^{2k+1}\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\right)\\
&\quad=-\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\right).
\end{aligned}$$

Hence, $\operatorname{tr}\left(\gamma_{i_1,2\widehat{n}+1}\cdots\gamma_{i_{2k+1},2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\gamma_{i,2\widehat{n}+1}\right)=0$.

The second assertion can be proved along the same lines.

The third assertion follows upon taking into account the cancellation and anti-commuting properties of the algebra in conjunction with the first statement, once the following equality has been established:

$$\operatorname{tr}\left(\gamma_{1,2\widehat{n}+1}\cdots\gamma_{2\widehat{n}+1,2\widehat{n}+1}\right)=(2i)^{\widehat{n}}.$$

To verify the latter identity one computes

$$\begin{aligned}
&\operatorname{tr}\left(\gamma_{1,2\widehat{n}+3}\cdots\gamma_{2\widehat{n}+3,2\widehat{n}+3}\right)\\
&\quad=\operatorname{tr}\left(\gamma_{1,2\widehat{n}+2}\cdots\gamma_{2\widehat{n}+2,2\widehat{n}+2}(-i)^{\widehat{n}+1}\gamma_{1,2\widehat{n}+2}\cdots\gamma_{2\widehat{n}+2,2\widehat{n}+2}\right)\\
&\quad=(-i)^{\widehat{n}+1}\operatorname{tr}\Big(\left(\sigma_1\circ\gamma_{1,2\widehat{n}}\right)\cdots\left(\sigma_1\circ\gamma_{2\widehat{n},2\widehat{n}}\right)i^{\widehat{n}}\left(\sigma_1\circ\gamma_{1,2\widehat{n}}\ldots\gamma_{2\widehat{n},2\widehat{n}}\right)\left(\sigma_2\circ I_{2^{\widehat{n}}}\right)\\
&\qquad\times\left(\sigma_1\circ\gamma_{1,2\widehat{n}}\right)\cdots\left(\sigma_1\circ\gamma_{2\widehat{n},2\widehat{n}}\right)i^{\widehat{n}}\left(\sigma_1\circ\gamma_{1,2\widehat{n}}\ldots\gamma_{2\widehat{n},2\widehat{n}}\right)\left(\sigma_2\circ I_{2^{\widehat{n}}}\right)\Big)\\
&\quad=\left(i^2\right)^{\widehat{n}}(-i)^{\widehat{n}+1}\operatorname{tr}\left(\sigma_1^{2\widehat{n}+1}\sigma_2\sigma_1^{2\widehat{n}+1}\sigma_2\circ\left(\gamma_{1,2\widehat{n}}\ldots\gamma_{2\widehat{n},2\widehat{n}}\right)^4\right)\\
&\quad=(-1)^{\widehat{n}+1}(-i)^{\widehat{n}+1}\operatorname{tr}\left(\sigma_1\sigma_1\sigma_2\sigma_2\circ I_{2^{\widehat{n}}}\right)=i^{\widehat{n}+1}2^{\widehat{n}+1}. \qquad\square
\end{aligned}$$

We conclude with the following result.

**Corollary A.9.** *Let $n\in\mathbb{N}_{\geqslant 2}$ be odd, $V$ be a complex vector space, $k\in\mathbb{N}_0$, with $k+1<n$, $i_1,\ldots,i_k\in\{1,\ldots,n\}$. Let $\Phi\colon\{1,\ldots,n\}^n\to V$ be satisfying the property*

$$\begin{aligned}
&\sum_{(i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k-2}}\Phi(i_1,\ldots,i_k,i,j,i_{k+3},\ldots,i_n)\\
&\quad=\sum_{(i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k-2}}\Phi(j_1,\ldots,j_k,j,i,i_{k+3},\ldots,i_n),\quad i,j\in\{1,\ldots,n\}.
\end{aligned}$$

*Then*

$$\sum_{(i_{k+1},i_{k+2},i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k}}\operatorname{tr}\left(\gamma_{i_1,n}\cdots\gamma_{i_n,n}\right)\Phi(i_1,\ldots,i_n)=0,$$

*where $\gamma_{j,n}$, $j\in\{1,\ldots,n\}$, are given by Definition* A.3.

*Proof.* In the course of this proof we shall suppress the index $n$ in $\gamma_{i,n}$.

$$\sum_{(i_{k+1},i_{k+2},i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k}}\gamma_{i_1}\cdots\gamma_{i_n}\Phi(i_1,\ldots,i_k,i_{k+1},i_{k+2},i_{k+3},\ldots,i_n)$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}} \gamma_{i_1} \cdots \gamma_{i_n} \Phi(i_1, \dots, i_k, i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \\
&\quad + \frac{1}{2} \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}} \gamma_{i_1} \cdots \gamma_{i_n} \\
&\qquad\qquad \times \Phi(i_1, \dots, i_k, i_{k+2}, i_{k+1}, i_{k+3}, \dots, i_n) \\
&= \frac{1}{2} \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}} \\
&\quad \big(\gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+1}} \gamma_{i_{k+2}} \gamma_{i_{k+3}} \cdots \gamma_{i_n} + \gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+2}} \gamma_{i_{k+1}} \gamma_{i_{k+3}} \cdots \gamma_{i_n,n}\big) \\
&\quad \times \Phi(i_1, \dots, i_n) \\
&= \frac{1}{2} \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}, i_{k+1} \neq i_{k+2}} \\
&\quad \big(\gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+1}} \gamma_{i_{k+2}} \gamma_{i_{k+3}} \cdots \gamma_{i_n} + \gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+2}} \gamma_{i_{k+1}} \gamma_{i_{k+3}} \cdots \gamma_{i_n}\big) \\
&\quad \times \Phi(i_1, \dots, i_n) \\
&\quad + \frac{1}{2} \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}, i_{k+1} = i_{k+2}} \\
&\quad \big(\gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+1}} \gamma_{i_{k+2}} \gamma_{i_{k+3}} \cdots \gamma_{i_n} + \gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+2}} \gamma_{i_{k+1}} \gamma_{i_{k+3}} \cdots \gamma_{i_n}\big) \\
&\quad \times \Phi(i_1, \dots, i_n) \\
&= \frac{1}{2} \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}, i_{k+1} = i_{k+2}} \\
&\quad \big(\gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+3}} \cdots \gamma_{i_n} + \gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+3}} \cdots \gamma_{i_n}\big) \Phi(i_1, \dots, i_n) \\
&= \sum_{(i_{k+1}, i_{k+2}, i_{k+3}, \dots, i_n) \in \{1,\dots,n\}^{n-k}, i_{k+1} = i_{k+2}} \gamma_{i_1} \cdots \gamma_{i_k} \gamma_{i_{k+3}} \cdots \gamma_{i_n} \Phi(i_1, \dots, i_n).
\end{aligned}
$$

Applying the internal trace to the latter sum, one infers that each term vanishes by Proposition A.8. $\square$

## Appendix B. A Counterexample to [22, Lemma 5]

In this appendix we shall provide a counterexample for the trace class property asserted in [22, Lemma 5]. The counterexample is constructed in dimension $n = 3$ and recorded in Theorem B.5.

Analogously to Example 4.8, we let $\Phi$ assume values in the $2 \times 2$ matrices and denote the Pauli matrices (see also Example 4.8) again by $\sigma_j$, $j \in \{1, 2, 3\}$. Before we give an explicit formula for $\Phi$, we need the following definitions. Let $\phi_1 \in C^\infty(\mathbb{R})$ be a function interpolating between 0 and 1 with

$$\phi_1(x) = \begin{cases} 0, & x \leqslant 0, \\ 1, & x \geqslant 1, \end{cases} \quad x \in \mathbb{R}, \quad \phi_2 \coloneqq \phi_1(-(\cdot + 1))$$

and let

$$\phi_{1,r,t} \coloneqq \phi_1\big(t^{-1}(\cdot) - t^{-1}r\big), \quad \phi_{2,r,t} \coloneqq \phi_2\big(t^{-1}(\cdot) - t^{-1}r\big), \quad r, t > 0.$$

For $r_1, r_2, t_1, t_2 \in (0, \infty)$ with $r_1 + t_1 < r_2 - t_2$, this yields the following variant of a smooth "cut-off" function

$$\psi_{r_1,r_2,t_1,t_2} \coloneqq \phi_{1,r_1,t_1} \phi_{2,r_2-t_2,t_2}. \tag{B.1}$$

One notes that $\psi_{r_1,r_2,t_1,t_2} \in C^\infty(\mathbb{R})$. We will use the following properties of $\psi_{r_1,r_2,t_1,t_2}$ (all of them are easily checked):

$$0 \leqslant \psi_{r_1,r_2,t_1,t_2} \leqslant 1, \tag{B.2}$$

$$\psi_{r_1,r_2,t_1,t_2}|_{[r_1+t_1, r_2-t_2]} = 1, \tag{B.3}$$

$$\psi_{r_1,r_2,t_1,t_2}|_{\mathbb{R} \setminus [r_1, r_2]} = 0, \tag{B.4}$$

$$|\psi'_{r_1,r_2,t_1,t_2}| \leqslant d_1 \left( \frac{1}{t_1} \vee \frac{1}{t_2} \right) \text{ on } [r_1, r_1 + t_1] \cup [r_2 - t_2, r_2], \tag{B.5}$$

$$|\psi^{(\ell)}_{r_1,r_2,t_1,t_2}| \leqslant d_\ell \left( \frac{1}{t_1^\ell} \vee \frac{1}{t_2^\ell} \right), \quad \ell \in \mathbb{N}_{\geqslant 2}, \tag{B.6}$$

with $d_1 \coloneqq \|\phi_1'\|_\infty \coloneqq \sup_{x \in \mathbb{R}} |\phi_1'(x)|$ and $d_\ell \coloneqq \|\phi_1^{(\ell)}\|_\infty$, $\ell \in \mathbb{N}_{\geqslant 2}$. For $k \in \mathbb{N}_{>1}$ define

$$r_k \coloneqq \sum_{j=1}^{k-1} 2^j = 2^k - 2,$$

$$\psi_{1,k} \coloneqq \psi_{r_k, r_{k+1}, \frac{1}{2}2^k, \frac{1}{20}2^k}, \quad \psi_{2,k} \coloneqq \psi_{r_k, r_{k+1}, \frac{1}{36}2^k, \frac{17}{18}2^k}.$$

One observes that

$$r_k + \frac{1}{2}2^k = r_{k+1} - \frac{1}{2}2^k < r_{k+1} - \frac{1}{20}2^k, \quad r_k + \frac{1}{36}2^k = r_{k+1} - \frac{35}{36}2^k < r_{k+1} - \frac{17}{18}2^k,$$

so that $\psi_{1,k}$ and $\psi_{2,k}$ are well-defined. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we let $\Phi \colon \mathbb{R}^3 \to \mathbb{C}^{2\times 2}$ be defined as follows,

$$\Phi(x) \coloneqq \sum_{j=1}^{3} \sigma_j + \sum_{k=2}^{\infty} \frac{1}{k^{1/3}} \sum_{j=1}^{3} \sigma_j \, \xi_{k,j}(x), \quad x \in \mathbb{R}^3, \tag{B.7}$$

where

$$\xi_{k,j}(x) \coloneqq \frac{1}{r_{k+1}} \psi_{1,k}(|x|)(x_j - r_k)\psi_{2,k}(x_j), \quad x \in \mathbb{R}^3. \tag{B.8}$$

One observes that $\Phi \in C^\infty(\mathbb{R})$. Next, we introduce the sets

$$B_k := \{x \in \mathbb{R}^3 \,|\, r_k \leqslant |x| \leqslant r_{k+1}\} \cap \bigcup_{j\in\{1,2,3\}} \{x \in \mathbb{R}^3 \,|\, r_k \leqslant x_j \leqslant r_{k+1}\}, \quad k \in \mathbb{N}, \tag{B.9}$$

and

$$\begin{aligned} \widetilde{B}_k := &\left\{x \in \mathbb{R}^3 \,\middle|\, r_k + \frac{1}{2}2^k \leqslant |x| \leqslant r_{k+1} - \frac{1}{20}2^k\right\} \\ &\bigcap \left\{x \in \mathbb{R}^3 \,\middle|\, r_k + \frac{1}{36}2^k \leqslant x_1, x_2, x_3 \leqslant r_{k+1} - \frac{17}{18}2^k\right\}, \quad k \in \mathbb{N}. \end{aligned} \tag{B.10}$$

Before turning to the properties of $\Phi$, we study $\xi_{k,j}$ first.

**Lemma B.1.** *Let $j \in \{1,2,3\}$, $\ell \in \{1,2,3\}$, $\xi_{k,j}$ as in* (B.8)*, $B_k$, $\widetilde{B}_k$ as in* (B.9) *and* (B.10)*, respectively. Then the following assertions* ($\alpha$)–($\gamma$) *hold:*

($\alpha$) *For all $k \in \mathbb{N}$, $x \in \mathbb{R}^3$, $\xi_{k,j}(x) \neq 0$ implies $x \in B_k$.* (B.11)

($\beta$) *For all $\alpha \in \mathbb{N}_0^3$, there exists $\kappa > 0$ such that for all $k \in \mathbb{N}$,*

$$|\partial^\alpha \xi_{k,j}(x)| \leqslant \kappa(1+|x|)^{-|\alpha|}, \quad x \in B_k. \tag{B.12}$$

($\gamma$) *For all $\ell \in \{1,2,3\}$, and all $k \in \mathbb{N}$, $\partial_\ell \xi_{k,j}(x) = \delta_{\ell j}$, $x \in \widetilde{B}_k$.* (B.13)

*Proof.* (B.11): The assertion follows from (B.4) and the definition of $B_k$.
(B.12): One observes that $\psi_{2,k} \neq 0$ on $(r_k, r_{k+1})$ and that $0 \leqslant \psi_{2,k} \leqslant 1$ by (B.4) and (B.2); hence,

$$|(x_j - r_k)\psi_{2,k}(x_j)| \leqslant 2^k, \quad j \in \{1,2,3\},\ k \in \mathbb{N}_{\geqslant 2}.$$

One recalls,

$$r_k = \sum_{j=1}^{k-1} 2^j = 2^k - 2 < r_{k+1} = 2^{k+1} - 2 = 2(2^k - 1),$$

in particular, $(1/r_{k+1}) \leqslant \kappa_0 2^{-k}$ for some $\kappa_0 > 0$. Hence,

$$\left\| \frac{1}{r_{k+1}} \psi_{1,k}(|x|) \sum_{j=1}^{3} \sigma_j (x_j - r_k)\psi_{2,k}(x_j) \right\| \leqslant \chi_{B_k}(x)\kappa_0, \quad x \in \mathbb{R}^3,$$

with $B_k$ introduced in (B.9). Thus, (B.12) holds for $\ell = 0$. Next, for the first partial derivatives in item (B.12) one computes for $\ell \neq j$,

$$(\partial_\ell \xi_{k,j})(x) = \frac{1}{r_{k+1}} \psi'_{1,k}(|x|) \frac{x_\ell}{|x|}(x_j - r_k)\psi_{2,k}(x_j)$$

and for $\ell = j$,

$$\begin{aligned} (\partial_j \xi_{k,j})(x) = &\frac{1}{r_{k+1}} \psi'_{1,k}(|x|) \frac{x_j}{|x|}(x_j - r_k)\psi_{2,k}(x_j) + \frac{1}{r_{k+1}} \psi_{1,k}(|x|)\sigma_j \psi_{2,k}(x_j) \\ &+ \frac{1}{r_{k+1}} \psi_{1,k}(|x|)\sigma_j (x_j - r_k)\psi'_{2,k}(x_j), \quad j \in \{1,2,3\}. \end{aligned}$$

For $x \in B_k$, one has $|(x_j - r_k)\psi'_k(x_j)| \leqslant c$ by (B.5), $|\psi'_k(|x|)(x_\ell - r_k)\psi_k(x_\ell)| \leqslant c$ by (B.2) and (B.5) and for some $\kappa, c > 0$ and all $k \in \mathbb{N}_{\geqslant 2}$,

$$\left\| \frac{1}{r_{k+1}} \psi_k(|x|)\sigma_j \psi_k(x_j) \right\| \leqslant \left\| \frac{1}{r_{k+1}} \psi_k(|x|) \right\| \leqslant \kappa(1+|x|)^{-1}$$

since for all $x \in B_k$ one has $r_{k+1} \geqslant |x|$. Higher-order derivatives can be treated similarly, using (B.6), proving assertion (B.12).

(B.13): This is obvious. □

The next lemma gives an account of the asymptotic properties of $\Phi$ and its derivatives.

**Lemma B.2.** *Let $\Phi$ be given by* (B.7)*. Then the following assertions* ($\alpha$)–($\gamma$) *hold:*

($\alpha$) *$\Phi$ is bounded, pointwise self-adjoint, $\Phi \in C^\infty\big(\mathbb{R}^3; \mathbb{C}^{2\times 2}\big)$, $\Phi(x)^{-1}$ exists for all $x \in \mathbb{R}^3$, and $\Phi(x)^2 \underset{|x|\to\infty}{\longrightarrow} I_2$.*

($\beta$) *There exists $\kappa > 0$ such that*

$$|(\partial_j \Phi)(x)| \leqslant \kappa(1+|x|)^{-1}, \quad x \in \mathbb{R}^3,\ j \in \{1,2,3\},$$

*and the formula*

$$(\partial_j \Phi)(x) = k^{-1/3}\sigma_j \quad x \in \widetilde{B}_k,\ j \in \{1,2,3\},\ k \in \mathbb{N},$$

*holds, where $\widetilde{B}_k$ is given by* (B.10)*.*

($\gamma$) *For all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 2$, there exists $\kappa' > 0$, such that*

$$|(\partial^\alpha \Phi)(x)| \leqslant \kappa' \left(1+|x|\right)^{-|\alpha|}, \quad x \in \mathbb{R}^3.$$

*Proof.* For item ($\alpha$), we use Lemma B.1 (B.12) with $\ell = 0$ together with the fact that $B_k \cap B_{k'} = \emptyset$ for $k' > k+1$, so $\Phi$ is bounded. $\Phi$ is easily verified to be pointwise self-adjoint. For showing invertibility of $\Phi$, one computes for $x \in B_k$,

$$\begin{aligned}
\Phi(x)\Phi(x) &= \left(\sum_{j=1}^{3}\left(\sigma_j + \frac{1}{k^{1/3}}\xi_{k,j}(x)\sigma_j\right)\right)^2 \\
&= \sum_{j=1}^{3}\left(1 + \frac{1}{k^{1/3}}\xi_{k,j}(x)\right)^2 I_2 \\
&= \sum_{j=1}^{3}\left(1 + 2\frac{1}{k^{1/3}}\xi_{k,j}(x) + \left(\frac{1}{k^{1/3}}\xi_{k,j}(x)\right)^2\right) I_2 \geqslant I_2,
\end{aligned}$$

implying ($\alpha$). Item ($\beta$) follows from Lemma B.1, (B.12), and (B.13), whereas item ($\gamma$) follows from (B.1), (B.12). □

In order to prove that $\operatorname{tr}_4\big((L^*L+z)^{-1} - (LL^*+z)^{-1}\big)$ for $L = \mathcal{Q} + \Phi$ (with $\mathcal{Q}$ as in (6.3)) is *not* trace class for $z$ in a neighborhood of 0, we need to invoke the following general statement:

**Theorem B.3** ([16, Theorem 3.1])**.** *Let $K \in \mathcal{B}(L^2(\mathbb{R}^n))$ be an operator induced by a continuous integral kernel $k\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. Assume that $K \in \mathcal{B}_1\big(L^2(\mathbb{R}^n)\big)$. Then the function $x \mapsto k(x,x)$ defines an element of $L^1(\mathbb{R}^n)$.*

Before we state and prove the main result of this section, we need to study the volume of $\widetilde{B}_k$:

**Lemma B.4.** *Let $\widetilde{B}_k$, $k \in \mathbb{N}$, be as in* (B.10)*. Then there exists $k_0 \in \mathbb{N}$, such that for all $k \in \mathbb{N}_{\geqslant k_0}$,*

$$\operatorname{vol}\big(\widetilde{B}_k\big) = 2^{3k}/(36)^3.$$

*Proof.* Let $k \in \mathbb{N}$. One observes that if

$$x \in \left\{ x \in \mathbb{R}^3 \,\middle|\, r_k + \frac{1}{36}2^k \leqslant x_1, x_2, x_3 \leqslant r_{k+1} - \frac{17}{18}2^k \right\},$$

then

$$\sqrt{3}\left(r_k + \frac{1}{36}2^k\right) \leqslant |x| \leqslant \sqrt{3}\left(r_{k+1} - \frac{17}{18}2^k\right).$$

Since $16/10 \leqslant \sqrt{3} \leqslant 18/10$, for sufficiently large $k \in \mathbb{N}$, the estimates

$$\sqrt{3}\left(r_k + \frac{1}{36}2^k\right) \geqslant \left(\frac{16}{10} + \frac{16}{10}\frac{1}{36}\right)2^k - \frac{16}{10}2 \geqslant \frac{3}{2}2^k - 2 = r_k + \frac{1}{2}2^k,$$

and

$$\sqrt{3}\left(r_{k+1} - \frac{17}{18}2^k\right) \leqslant \frac{18}{10}\left(2 - \frac{17}{18}\right)2^k - 2\frac{18}{10} \leqslant \frac{19}{10}2^k - 2 = r_{k+1} - \frac{1}{20}2^k,$$

hold. Consequently, for sufficiently large $k \in \mathbb{N}$,

$$\left\{ x \in \mathbb{R}^3 \,\middle|\, r_k + \frac{1}{36}2^k \leqslant x_1, x_2, x_3 \leqslant r_{k+1} - \frac{17}{18}2^k \right\} \subseteq \widetilde{B}_k.$$

Hence, there exists $k_0 \in \mathbb{N}$, such that for all $k \in \mathbb{N}_{\geqslant k_0}$,

$$\operatorname{vol}\left(\widetilde{B}_k\right) = \left(r_{k+1} - \frac{17}{18}2^k - (r_k + \frac{1}{36}2^k)\right)^3. \qquad \square$$

**Theorem B.5.** *Let $n = 3$ and $\mathcal{Q}$ and $\Phi$ be given by* (6.3) *and* (B.7)*, respectively. Then there exists $\delta > 0$ such that for $L = \mathcal{Q} + \Phi$, and for any real $z \in B(0, \delta)\backslash\{0\}$,*

$$\operatorname{tr}_4\left((L^*L + z)^{-1} - (LL^* + z)^{-1}\right) \notin \mathcal{B}_1\big(L^2(\mathbb{R}^3)\big).$$

*Proof.* In view of Remark 11.3 and Lemma 7.7 it suffices to check whether or not

$$\widetilde{T} \coloneqq \operatorname{tr}_4\left((R_{1+z}C)^3 R_{1+z}\right)$$

is a trace class operator, where $C = [\mathcal{Q}, \Phi]$, and $R_{1+z}$ are given by (2.2) and (4.6), respectively.

Arguing by contradiction, we shall assume that $\widetilde{T} \in \mathcal{B}_1\big(L^2(\mathbb{R}^3)\big)$. One observes,

$$\begin{aligned}
(R_{1+z}C)^3 R_{1+z} &= R_{1+z}CR_{1+z}CR_{1+z}CR_{1+z} \\
&= [R_{1+z}, C]R_{1+z}CR_{1+z}CR_{1+z} + CR_{1+z}R_{1+z}CR_{1+z}CR_{1+z} \\
&= [R_{1+z}, C]R_{1+z}CR_{1+z}CR_{1+z} + CR_{1+z}[R_{1+z}, C]R_{1+z}CR_{1+z} \\
&\quad + CR_{1+z}CR_{1+z}R_{1+z}CR_{1+z} \\
&= [R_{1+z}, C]R_{1+z}CR_{1+z}CR_{1+z} + CR_{1+z}[R_{1+z}, C]R_{1+z}CR_{1+z} \\
&\quad + CR_{1+z}CR_{1+z}[R_{1+z}, C]R_{1+z} + CR_{1+z}CR_{1+z}CR_{1+z}R_{1+z}.
\end{aligned} \tag{B.14}$$

By Lemmas 4.5 and B.2, one gets $CR_{1+z}, R_{1+z}C \in \mathcal{B}_4\big(L^2(\mathbb{R}^3)\big)$ and $[R_{1+z}, C] \in \mathcal{B}_2\big(L^2(\mathbb{R}^3)\big)$. Hence, by Theorem 4.2, one infers that despite the last term in (B.14), all operators are trace class. In addition, one computes

$$\begin{aligned}
CR_{1+z}CR_{1+z}CR_{1+z}R_{1+z} &= C[R_{1+z}, C]R_{1+z}CR_{1+z}R_{1+z} \\
&\quad + C^2R_{1+z}R_{1+z}CR_{1+z}R_{1+z}
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
&= C[R_{1+z}, C]R_{1+z}CR_{1+z}R_{1+z} + C^2R_{1+z}[R_{1+z}, C]R_{1+z}R_{1+z} \\
&\quad + C^2R_{1+z}CR_{1+z}^3
\end{aligned}$$

$$= C[R_{1+z}, C]R_{1+z}CR_{1+z}R_{1+z} + C^2R_{1+z}[R_{1+z}, C]R_{1+z}R_{1+z}$$
$$+ C^2[R_{1+z}, C]R_{1+z}^3 + C^3R_{1+z}^4. \tag{B.16}$$

Next, one notes that Lemma 4.4 implies the relation

$$[R_{1+z}, C] = R_{1+z}\,(\Delta C)\,R_{1+z} + 2R_{1+z}\,(\mathcal{Q}C)\,\mathcal{Q}R_{1+z}.$$

With the help of Lemma B.2, there exists $\kappa > 0$ such that

$$\max\{\|C(x)^2\|, \|(\Delta C)(x)\|, \|(\mathcal{Q}C)(x)\|\} \leqslant \kappa(1 + |x|)^{-2}, \quad x \in \mathbb{R}^3.$$

Therefore, Lemma 4.5 and Theorem 4.2 imply

$$\begin{aligned} C[R_{1+z}, C]R_{1+z}CR_{1+z} &= C(R_{1+z}(\Delta C)R_{1+z} \\ &\qquad + 2R_{1+z}(\mathcal{Q}C)\mathcal{Q}R_{1+z})R_{1+z}CR_{1+z} \\ &= CR_{1+z}(\Delta C)R_{1+z}CR_{1+z} + 2CR_{1+z}\,(\mathcal{Q}C)\,R_{1+z}\mathcal{Q}R_{1+z}CR_{1+z} \\ &\in \mathcal{B}_4 \cdot \mathcal{B}_2 \cdot \mathcal{B}_4 + \mathcal{B}_4 \cdot \mathcal{B}_2 \cdot \mathcal{B} \cdot \mathcal{B}_4 \subseteq \mathcal{B}_1, \end{aligned}$$

and,

$$C^2R_{1+z}[R_{1+z}, C]R_{1+z} \in \mathcal{B}_2 \cdot \mathcal{B}_2 \cdot \mathcal{B} \subseteq \mathcal{B}_1,$$

as well as,

$$\begin{aligned} C^2[R_{1+z}, C]R_{1+z}^3 &= C^2R_{1+z}\,((\Delta C)\,R_{1+z} + 2R_{1+z}\,(\mathcal{Q}C)\,\mathcal{Q}R_{1+z})\,R_{1+z}^3 \\ &= C^2R_{1+z}\,(\Delta C)\,R_{1+z}R_{1+z}^3 + 2C^2R_{1+z}R_{1+z}\,(\mathcal{Q}C)\,QR_{1+z}R_{1+z}^3 \\ &\in \mathcal{B}_2 \cdot \mathcal{B}_2 \cdot \mathcal{B} + \mathcal{B}_2 \cdot \mathcal{B}_2 \cdot \mathcal{B} \subseteq \mathcal{B}_1. \end{aligned}$$

Noting that the inner trace maps trace class operators to trace class operators (cf. Remark 3.2), and combining (B.14) and (B.16) together with our assumption that $\widetilde{T}$ is trace class, one concludes that

$$T := \operatorname{tr}_4\left(C^3R_{1+z}^4\right) = \operatorname{tr}_4\left(C^3\right)R_{1+z}^4 \in \mathcal{B}_1\big(L^2(\mathbb{R}^3)\big).$$

Next, one observes that $T$ is an integral operator induced by the following integral kernel

$$t\colon (x, y) \mapsto \int_{(\mathbb{R}^3)^3} \operatorname{tr}_4\left(C^3\right)(x)r_{1+z}(x - x_1)r_{1+z}(x_1 - x_2)r_{1+z}(x_2 - x_3)r_{1+z}(x_3 - y) \\ \times d^3x_1d^3x_2d^3x_3,$$

where $r_{1+z}$ is the Helmholtz Green's function, see (5.11) associated with $(-\Delta + (1 + z))^{-1}$. By Theorem 5.1 (and Proposition 5.4), $t$ is continuous. As $T$ is trace class, Theorem B.3 implies that the map $x \mapsto t(x, x)$ generates an $L^1(\mathbb{R}^3)$-function. Hence,

$$\begin{aligned} &\int_{\mathbb{R}^3} |t(x, x)|\,d^3x \\ &\quad = \int_{\mathbb{R}^3} \bigg| \int_{(\mathbb{R}^3)^3} \operatorname{tr}_4\left(C^3\right)(x)r_{1+z}(x - x_1)r_{1+z}(x_1 - x_2)r_{1+z}(x_2 - x_3)r_{1+z}(x_3 - x) \\ &\qquad\qquad \times d^3x_1d^3x_2d^3x_3\bigg|\,d^3x \\ &\quad = \int_{\mathbb{R}^3} \bigg| \int_{(\mathbb{R}^3)^3} \operatorname{tr}_4\left(C^3\right)(x)r_{1+z}(x_1)r_{1+z}(x_1 - x_2)r_{1+z}(x_2 - x_3)r_{1+z}(x_3) \\ &\qquad\qquad \times d^3x_1d^3x_2d^3x_3\bigg|\,d^3x \end{aligned}$$

$$= \int_{\mathbb{R}^3} \left| \operatorname{tr}_4 \left(C^3\right)(x)\right| d^3x \left\langle \delta_{\{0\}}, R_{1+z}^4 \delta_{\{0\}} \right\rangle < \infty.$$

In other words,

$$\operatorname{tr}_4 \left(C^3\right) \in L^1(\mathbb{R}^3). \tag{B.17}$$

The rest of the proof aims at showing that the statement (B.17) is false. For this purpose we need to compute $\operatorname{tr}_4 \left([\mathcal{Q}, \Phi]^3\right)$ on $\bigcup_{k\in\mathbb{N}_{\geqslant 2}} \widetilde{B}_k$, with $\widetilde{B}_k$ given in (B.10). We recall from Lemma B.2 $(ii)$,

$$(\partial_j \Phi)(x) = \frac{1}{k^{1/3}} \frac{1}{r_{k+1}} \sigma_j, \quad x \in \widetilde{B}_k,\ j \in \{1, 2, 3\}.$$

Hence,

$$\begin{aligned}
\operatorname{tr}_4 \left([\mathcal{Q}, \Phi]^3\right)(x) &= \sum_{j,m,\ell=1}^{3} 2i\varepsilon_{jm\ell} \operatorname{tr}_2 \left((\partial_j \Phi)(x)(\partial_m \Phi)(x)(\partial_\ell \Phi)(x)\right) \\
&= \sum_{j,m,\ell=1}^{3} 2i\varepsilon_{jm\ell} \frac{1}{k} \frac{1}{r_{k+1}^3} \operatorname{tr}_2 \left(\sigma_j \sigma_m \sigma_\ell\right) \\
&= -\sum_{j,m,\ell=1}^{3} 4\varepsilon_{jm\ell}^2 \frac{1}{k} \frac{1}{r_{k+1}^3} \\
&= -24 \frac{1}{k} \frac{1}{r_{k+1}^3},
\end{aligned}$$

implying,

$$\left| \operatorname{tr}_4 \left([\mathcal{Q}, \Phi]^3\right)(x)\right| \geqslant 24 \frac{1}{k} \frac{1}{r_{k+1}^3}, \quad x \in \widetilde{B}_k,\ k \in \mathbb{N}_{\geqslant 2}. \tag{B.18}$$

However, employing Lemma B.4 one infers with the help (B.18) that for some $k_0 \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{tr}_4 \left(C^3\right) = \left\| \operatorname{tr}_4 \left([\mathcal{Q}, \Phi]^3\right)\right\|_{L^1(\mathbb{R}^3)} &\geqslant \sum_{k=k_0}^{\infty} \frac{1}{k} \frac{1}{r_{k+1}^3} \operatorname{vol}\left(\widetilde{B}_k\right) \\
&= \frac{1}{(36)^3} \sum_{k=k_0}^{\infty} \frac{1}{k} \frac{1}{r_{k+1}^3} 2^{3k} = \frac{1}{(36)^3} \sum_{k=k_0}^{\infty} \frac{1}{k} \frac{1}{\left(2^k - 2\right)^3} 2^{3k} = \infty,
\end{aligned}$$

contradicting (B.17). $\square$

*Remark* B.6. It might be of interest to compute the index of $\mathcal{Q}+\Phi$, with the potential $\Phi$ constructed in this section: One notes that $\Phi$ is a $\mathcal{Q}$-compact perturbation of the operator

$$\mathcal{Q} + U \text{ in } L^2(\mathbb{R}^n), \text{ where } U \coloneqq \sum_{j=1}^{3} \sigma_j.$$

Since $U^2 = I_2$ and $\partial_j U = 0$, $j \in \{1, 2, 3\}$, one infers that $U$ is admissible. The index formula in Theorem 10.1 leads to $\operatorname{ind}(\mathcal{Q} + U) = 0$, and hence to $\operatorname{ind}(\mathcal{Q} + \Phi) = 0$.

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